IDEOLOG: A PROGRAM FOR FILTERING ECONOMETRIC DATA—A SYNOPSIS OF ALTERNATIVE METHODS

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An account is given of various filtering procedures that have been implemented in a computer program, which can be used in analysing econometric time series. The program provides some new filtering procedures that operate primarily in the frequency domain. Their advantage is that they are able to achieve clear separations of components of the data that reside in adjacent frequency bands in a way that the conventional time-domain methods cannot.

Several procedures that operate exclusively within the time domain have also been implemented in the program. Amongst these are the bandpass filters of Baxter and King and of Christiano and Fitzgerald, which have been used in estimating business cycles. The Henderson filter, the Butterworth filter and the Leser or Hodrick–Prescott filter are also implemented. These are also described in this paper

Econometric filtering procedures must be able to cope with the trends that are typical of economic time series. If a trended data sequence has been reduced to stationarity by differencing prior to its filtering, then the filtered sequence will need to be re-inflated. This can be achieved within the time domain via the summation operator, which is the inverse of the difference operator. The effects of the differencing can also be reversed within the frequency domain by recourse to the frequency-response function of the summation operator.

1. Introduction

This paper gives an account of some of the facilities that are available in a new computer program, which implements various filters that can be used for extracting the components of an economic data sequence and for producing smoothed and seasonally-adjusted data from monthly and quarterly sequences.

The program can be downloaded from the following web address:

`http://www.le.ac.uk/users/dsgp1/`

It is accompanied by a collection of data and by three log files, which record steps that can be taken in processing some typical economic data. Here, we give an account of the theory that lies behind some of the procedures of the program.

The program originated in a desire to compare some new methods with existing procedures that are common in econometric analyses. The outcome has been a comprehensive facility, which will enable a detailed investigation of univariate econometric time series. The program will also serve to reveal the extent to which the results of an economic analysis might be the consequence of the choice of a particular filtering procedure.

The new procedures are based on the Fourier analysis of the data, and they perform their essential operations in the frequency domain as opposed to the time domain. They depend upon a Fourier transform for carrying the data into the frequency domain and upon an inverse transform for carrying the filtered elements back to the time domain. Filtering procedures usually operate exclusively in the time domain. This is notwithstanding fact that, for a proper understanding of the effects of a filter, one must know its frequency-response function.

The sections of this paper give accounts of the various classes of filters that have been implemented in the program. In the first category, to which section 2 is devoted, are the simple finite impulse response (FIR) or linear moving-average filters that endeavour to provide approximations to the so-called ideal frequency-selective filters. Also in this category of FIR filters is the time-honoured filter of Henderson (1916), which is part of a seasonal-adjustment program that is widely used in central statistical agencies.

The second category concerns filters of the infinite impulse response (IIR) variety, which involve an element of feedback. The filters of this category that are implemented in the program are all derived according to the Wiener–Kolmogorov principle. The principle has been enunciated in connection with the filtering of stationary and doubly-infinite data sequences—see Whittle (1983), for example. However, the purpose of the program is to apply these filters to short non stationary sequences. In section 3, the problem of non stationarity is broached, whereas, in section 4, the adaptations that are appropriate to short sequences are explained.

Section 5 deals with the new frequency-domain filtering procedures. The details of their implementation are described and some of their uses are highlighted. In particular, it is shown how these filters can achieve an ideal frequency selection, whereby all of the elements of the data that fall below a given cut-off frequency are preserved and all those that fall above it are eliminated.

2. The FIR filters

One of the purposes in filtering economic data sequences is to obtain a representation of the business cycle that is free from the distractions of seasonal fluctuations and of high-frequency noise. According to Baxter and King (1999), the business cycle should comprise all elements of the data that have cyclical durations of no less than of one and a half years and not exceeding eight years. For this purpose, they have proposed to use a moving-average bandpass filter to approximate the ideal frequency-selective filter. An alternative approximation, which has the same purpose, has been proposed by Christiano and Fitzgerald (2003). Both of these filters have been implemented in the program.

A stationary data sequence can be resolved into a sum of sinusoidal elements whose frequencies range from zero up to the Nyquist frequency of π radians per sample interval, which represents the highest frequency that is observable in sam-

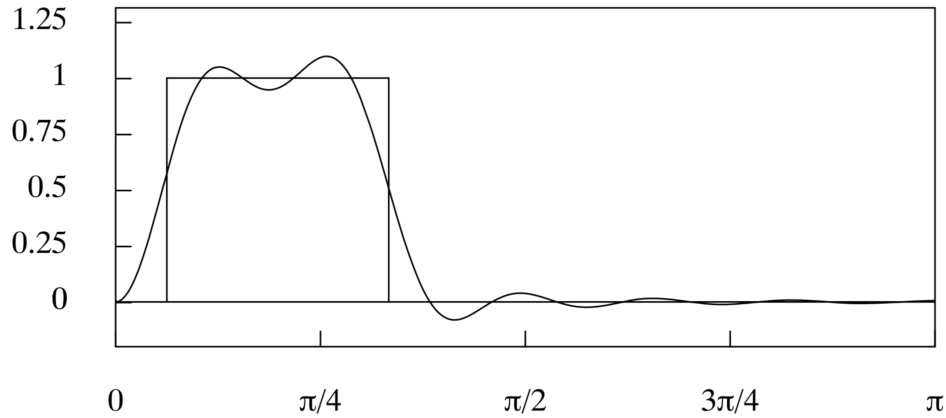

Figure 1. The frequency response of the truncated bandpass filter of 25 coefficients superimposed upon the ideal frequency response. The lower cut-off point is at $\pi/16$ radians (11.25°), corresponding to a period of 32 quarters, and the upper cut-off point is at $\pi/3$ radians (60°), corresponding to a period of the 6 quarters.

pled data. A data sequence $\{y_t, t = 0, 1, \ldots, T-1\}$ comprising $T = 2n$ observations has the following Fourier decomposition:

$$y_t = \sum_{j=0}^{n} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\}. \tag{1}$$

Here, $\omega_j = 2\pi j/T; j = 0, \ldots, n$, are the Fourier frequencies, which are equally spaced in the interval $[0, \pi]$, whereas α_j, β_j are the associated Fourier coefficients, which indicate the amplitudes of the sinusoidal elements of the data sequence. An ideal filter is one that transmits the elements that fall within a specified frequency band, described as the pass band, and which blocks elements at all other frequencies, which constitute the stop band.

In representing the properties of a linear filter, it is common to imagine that it is operating on a doubly-infinite data sequence of a statistically stationary nature. Then, the Fourier decomposition comprises an infinity of sinusoidal elements of negligible amplitudes whose frequencies form a continuum in the interval $[0, \pi]$. The frequency-response function of the filter displays the factors by which the amplitudes of the elements are altered in their passage through the filter.

For an ideal filter, the frequency response is unity within the pass band and zero within the stop band. Such a response is depicted in Figure 1, where the pass band, which runs from $\pi/16$ to $\pi/3$ radians per sample interval, is intended to transmit the elements of a quarterly econometric data sequence that constitute the business cycle.

To achieve an ideal frequency selection with a linear moving-average filter would require an infinite number of filter coefficients. This is clearly impractical;

and so the sequence of coefficients must be truncated, whereafter it may be modified in certain ways to diminish the adverse effects of the truncation.

Approximation to the Ideal Filter

Figure 1 also shows the frequency response of a filter that has been derived by taking twenty-five of the central coefficients of the ideal filter and adjusting their values by equal amounts so that they sum to zero. This is the filter that has been proposed by Baxter and King (1999) for the purpose of extracting the business cycle from economic data. The filter is affected by a considerable leakage, whereby elements that fall within the stop band are transmitted in part by the filter.

The z-transform of a sequence $\{\psi_j\}$ of filter coefficients is the polynomial $\psi(z) = \sum_j \psi_j z$. Constraining the coefficients to sum to zero ensures that the polynomial has a root of unity, which is to say that $\psi(1) = \sum_j \psi_j = 0$. This implies that $\nabla(z) = 1 - z$ is a factor of the polynomial, which indicates that the filter incorporates a differencing operator.

If the filter is symmetric, such that $\psi(z) = \psi_0 + \psi_1(z + z^{-1}) + \cdots + \psi_q(z^q + z^{-q})$ and, therefore, $\psi(z) = \psi(z^{-1})$, then $1 - z^{-1}$ is also a factor. Then, $\psi(z)$ has the combined factor $(1 - z)(1 - z^{-1}) = -z\nabla(z)^2$, which indicates that the filter incorporates a twofold differencing operator. Such a filter is effective in reducing a linear trend to zero; and, therefore, it is applicable to econometric data sequences that have an underlying log-linear tend.

The filter of Baxter and King (1999), which fulfils this condition, is appropriate for the purpose of extracting the business cycle from a trended data sequence. Figure 2 shows the logarithms of data of U.K. real domestic consumption for the years 1955–1994 through which a linear trend has been interpolated. Figure 3 shows the results of subjecting these data to the Baxter–King filter. A disadvantage of the filter, which is apparent in Figure 3, is that it is incapable of reaching the ends of the sample. The first q sample values and the last q remain unprocessed.

To overcome this difficulty, Christiano and Fitzgerald (2003) have proposed a filter with a variable set of coefficients. To generate the filtered value at time t, they associate the central coefficient ψ_0 with y_t. If y_{t-p} falls within the sample, then they associate it with the coefficient ψ_p. Otherwise, if it falls outside the sample, it is disregarded. Likewise, if y_{t+p} falls within the sample, then it is associated with ψ_p, otherwise it is disregarded. If the data follow a first-order random walk, then the first and the last sample elements y_0 and y_{T-1} receive extra weights A and B, which correspond to the sums of the coefficients discarded from the filter at either end. The resulting filtered value at time t may be denoted by

$$\begin{aligned} x_t = Ay_0 + \psi_t y_0 + \cdots + \psi_1 y_{t-1} + \psi_0 y_t \\ + \psi_1 y_{t+1} + \cdots + \psi_{T-1-t} y_{T-1} + By_{T-1}. \end{aligned} \tag{2}$$

This equation comprises the entire data sequence $y_0, \ldots, y_{T-1}$; and the value of t determines which of the coefficients of the infinite-sample filter are involved

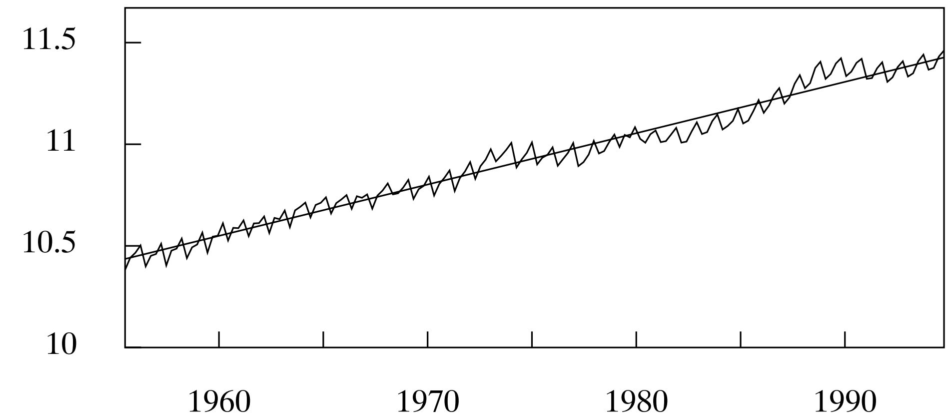

Figure 2. The quarterly sequence of the logarithms of consumption in the U.K., for the years 1955 to 1994, together with a linear trend interpolated by least-squares regression.

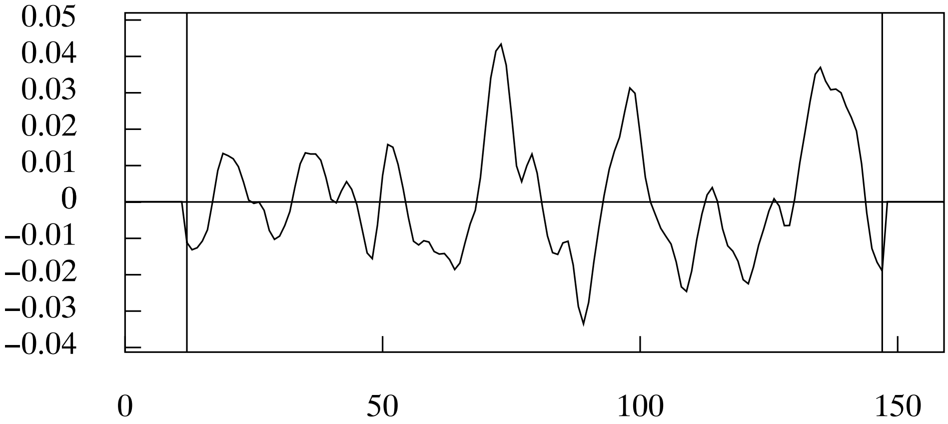

Figure 3. The sequence derived by applying the truncated bandpass filter of 25 coefficients to the quarterly logarithmic data on U.K. Consumption.

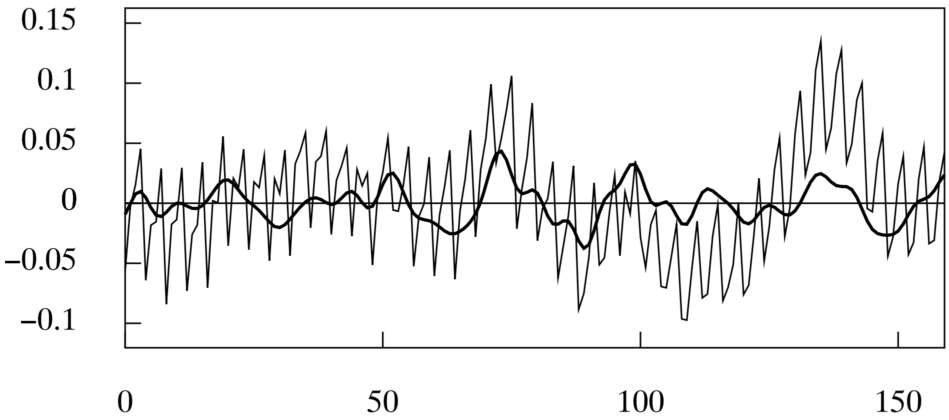

Figure 4. The sequence derived by applying the bandpass filter of Christiano and Fitzgerald to the quarterly logarithmic data on U.K. Consumption.

in producing the current output. Thus, the value of x_0 is generated by looking forwards to the end of the sample, whereas the value of x_{T-1} is generated by looking backwards to the beginning of the sample.

For data that appear to have been been generated by a first-order random walk with a constant drift, it is appropriate to extract a linear trend before filtering the residual sequence. Figure 4 provides an example of the practice. In fact, this has proved to be the usual practice in most circumstances.

Within the category of FIR filters, the program also implements the time honoured smoothing filter of Henderson (1916), which forms an essential part of the detrending procedure of the X-11 program of the Bureau of the Census. This program provides the method of seasonal adjustment that is used predominantly by central statistical agencies.

Here, the end-of-sample problem is overcome by supplementing the Henderson filter with a set of asymmetric filters that can be applied to the elements of the first and the final segments. These are the Musgrave (1964) filters. (See Quenneville, Ladiray and Lefranc, 2003 for a recent account of these filters.) In the X-11 ARIMA variant, which is used by Statistics Canada, the alternative recourse is adopted of extrapolating the data beyond the ends of the sample so that it can support a time-invariant filter that does run to the ends.

3. The Wiener–Kolmogorov Filters

The program also provides several filters of the feedback variety that are commonly described as infinite-impulse response (IIR) filters. The filters in question are derived according to the finite-sample Wiener–Kolmogorov principle that has been expounded by Pollock (2000, 2007).

The ordinary theory of Wiener–Kolmogorov filtering assumes a doubly-infinite data sequence $y(t) = \xi(t) + \eta(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ generated by a stationary stochastic process. The process is compounded from a signal process $\xi(t)$ and a noise process $\eta(t)$ that are assumed to be statistically independent and to have zero-valued means. Then, the autocovariance generating function of $y(t)$ is given by

$$\gamma_y(z) = \gamma_\xi(z) + \gamma_\eta(z), \tag{3}$$

which is sum of the autocovariance functions of $\xi(t)$ and $\eta(t)$.

The object is to extract estimates of the signal sequence $\xi(t)$ and noise sequence $\eta(t)$ from the data sequence. The z-transforms of the relevant filters are

$$\beta_\xi(z) = \frac{\gamma_\xi(z)}{\gamma_\xi(z) + \gamma_\eta(z)} = \frac{\psi_\xi(z^{-1})\psi_\xi(z)}{\phi(z^{-1})\phi(z)}, \tag{4}$$

and

$$\beta_\eta(z) = \frac{\gamma_\eta(z)}{\gamma_\xi(z) + \gamma_\eta(z)} = \frac{\psi_\eta(z^{-1})\psi_\eta(z)}{\phi(z^{-1})\phi(z)}. \tag{5}$$

It can been that $\beta_\xi(z) + \beta_\eta(z) = 1$, in view of which the filters can be described as complementary.

The factorisations of the filters that are given on the RHS enable them to be applied via a bi-directional feedback process. In the case of the signal extraction filter $\beta_\xi(z)$, the process in question can be represented by the equations

$$\phi(z)q(z) = \psi_\xi(z)y(z) \qquad \text{and} \qquad \phi(z^{-1})x(z) = \psi_\xi(z^{-1})q(z^{-1}), \tag{6}$$

wherein $q(z)$, $y(z)$ and $x(z)$ stand for the z-transforms of the corresponding sequences $q(t)$, $y(t)$ and $x(t)$.

To elucidate these equations, we may note that, in the first of them, the expression associated with z^t is

$$\sum_{j=0}^{m} \phi_j q_{t-j} = \sum_{j=0}^{n} \psi_{\xi,j} y_{t-j}. \tag{7}$$

Given that $\phi_0 = 1$, this serves to determine the value of q_t. Moreover, given that the recursion is assumed to be stable, there need be no restriction on the range of t. The first equation, which runs forward in time, generates an intermediate output $q(t)$. The second equation, which runs backwards in time, generates the final filtered output $x(t)$.

Filters for Trended Data

The classical Wiener–Kolmogorov theory can be extended in a straightforward way to cater for non stationary data generated by integrated autoregressive moving-average (ARIMA) processes in which the autoregressive polynomial contains roots of unit value. Such data processes can be described by the equation

$$y(z) = \frac{\delta(z)}{\nabla^p(z)} + \eta(z) \quad \text{or, equivalently,} \quad \nabla^p(z)y(z) = \delta(z) + \nabla^p(z)\eta(z), \tag{8}$$

where $\delta(z)$ and $\eta(z)$ are, respectively, the z-transforms of the mutually independent stationary stochastic sequences $\delta(t)$ and $\eta(t)$, and where $\nabla^p(z) = (1 - z)^p$ is the p-th power of the difference operator.

Here, there has to be some restriction on the range of t together with the condition that the elements δ_t and η_t are finite within this range. Also, the z-transform must comprise the appropriate initial conditions, which are effectively concealed by the notation. (See Pollock 2008 on this point.)

Within the program, two such filters have been implemented. The first is the filter of Leser (1961) and of Hodrick and Prescott (1980, 1997), which is designed to extract the non stationary signal or trend component when the data are generated according to the equation

$$\nabla^2(z)y(z) = g(z) = \delta(z) + \nabla^2(z)\eta(z), \tag{9}$$

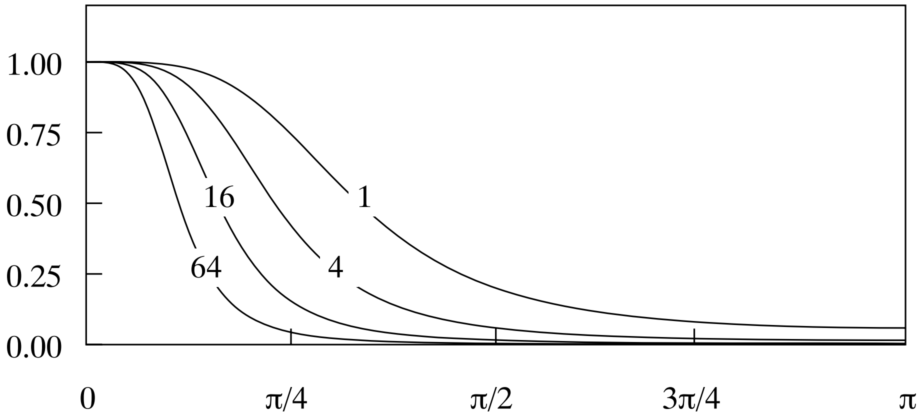

Figure 5. The frequency-response function of the Hodrick–Prescott smoothing filter for various values of the smoothing parameter λ.

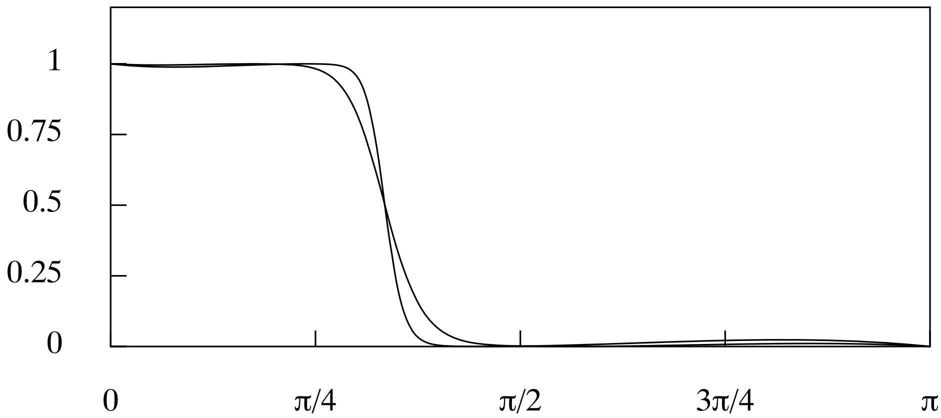

Figure 6. The frequency-response function of the lowpass Butterworth filters of orders $n = 6$ and $n = 12$ with a nominal cut-off point of $2\pi/3$ radians.

where $\delta(t)$ are $\eta(t)$ are mutually independent sequences of independently and identically distributed random variables, generated by so-called white-noise processes. With $\gamma_\delta(z) = \sigma_\delta^2$ and $\gamma_\xi(z) = \sigma_\delta^2\nabla(z^{-1})\nabla(z)$ and with $\gamma_\eta(z) = \sigma_\eta^2$, the z-transforms of the relevant filters become

$$\beta_\xi(z) = \frac{1}{1 + \lambda\nabla^2(z^{-1})\nabla^2(z)}, \tag{10}$$

and

$$\beta_\eta(z) = \frac{\nabla^2(z^{-1})\nabla^2(z)}{\lambda^{-1} + \nabla^2(z^{-1})\nabla^2(z)}, \tag{11}$$

where $\lambda = \sigma_\eta^2/\sigma_\delta^2$, which is described as the smoothing parameter.

The frequency-response functions of the filters for various values of λ are shown in Figure 5. These are obtained by setting $z = e^{-\mathrm{i}\omega} = \cos(\omega) - \mathrm{i}\sin(\omega)$ in the

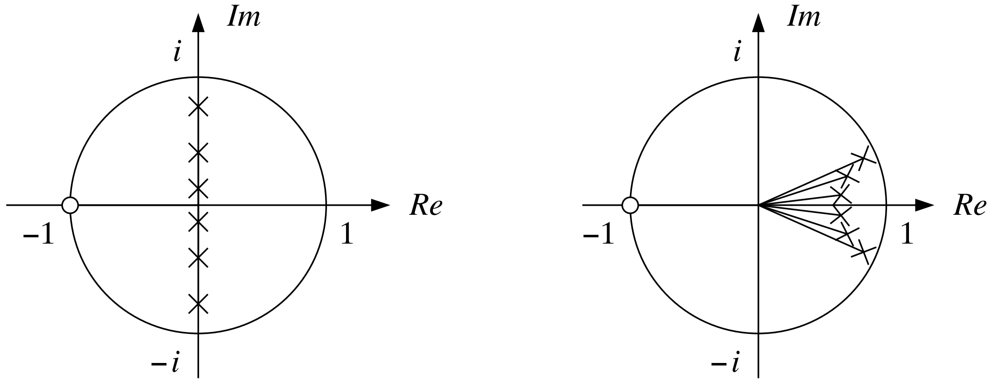

Figure 7. The pole–zero diagrams of the lowpass Butterworth filters for $n = 6$ when the cut-off is at $\omega = \pi/2$ (left) and at $\omega = \pi/8$.

formula of (10) and by letting ω run from 0 to π. (In the process, the imaginary quantities are cancelled so as to give rise to the real-valued functions that are plotted in the diagram.)

It is notable that the specification of the underlying process $y(t)$, in which both the signal component $\xi(z) = \delta(z)/\nabla(z)$ and the noise component $\eta(z)$ have spectral density functions that extend over the entire frequency range, precludes the clear separation of the components. This is reflected in the fact that, for all but the highest values λ, the filter transmits significant proportions of the elements at all frequencies.

The second of the Wiener–Kolmogorov filters that are implemented in the program is capable of a much firmer discrimination between the signal and noise than is the Leser (1961) filter. This is the Butterworth (1930) filter, which was originally devised as an analogue filter but which can also be rendered in digital form—See Pollock (2000). The filter is appropriate for extracting the component $(1+z)^n\delta(z)$ from the sequence

$$g(z) = (1+z)^n\delta(z) + (1-z)^n\kappa(z). \tag{12}$$

Here, $\delta(t)$ and $\kappa(t)$ denote independent white-noise processes, whereas there is usually $g(z) = \nabla^2(z)y(z)$, where $y(t)$ is the data process. This corresponds to the case where twofold differencing is required to eliminate a trend from the data. Under these circumstances, the equation of the data process is liable to be represented by

$$\begin{aligned} y(z) &= \xi(z) + \eta(z) \\ &= \frac{(1+z)^n}{\nabla^2(z)}\delta(z) + (1-z)^{n-2}\kappa(z). \end{aligned} \tag{13}$$

However, regardless of the degree of differencing to which $y(t)$ must be subjected in reducing it to stationarity, the z-transforms of the complementary filters will be

$$\beta_\xi(z) = \frac{(1+z^{-1})^n(1+z)^n}{(1+z^{-1})^n(1-z)^n + \lambda(1-z^{-1})^n(1+z)^n}, \tag{14}$$

and

$$\beta_\eta(z) = \frac{(1-z^{-1})^n(1-z)^n}{(1+z^{-1})^n(1-z)^n + \lambda^{-1}(1-z^{-1})^n(1+z)^n}, \tag{15}$$

where $\lambda = \sigma_\kappa^2/\sigma_\delta^2$.

It is straightforward to determine the value of λ that will place the cut-off of the filter at a chosen point $\omega_c \in (0, \pi)$. Consider setting $z = \exp\{-\mathrm{i}\omega\}$ in the formula of (14) of the lowpass filter. This gives the following expression for the gain:

$$\begin{aligned}\beta_\xi(e^{-\mathrm{i}\omega}) &= \frac{1}{1+\lambda\left(\mathrm{i}\dfrac{1-e^{-\mathrm{i}\omega}}{1+e^{-\mathrm{i}\omega}}\right)^{2n}} \\ &= \frac{1}{1+\lambda\big\{\tan(\omega/2)\big\}^{2n}}.\end{aligned} \tag{16}$$

At the cut-off point, the gain must equal 1/2, whence solving the equation $\beta_\xi(\exp\{-\mathrm{i}\omega_c\}) = 1/2$ gives $\lambda = \{1/\tan(\omega_c/2)\}^{2n}$.

Figure 6 shows how the rate of the transition of the Butterworth frequency response between the pass band and the stop band is affected by the order of the filter. Figure 7 shows the pole–zero diagrams of filters with different cut-off points. As the cut-off frequency is reduced, the transition between the two bands becomes more rapid. Also, some of the poles of the filter move towards the perimeter of the unit circle.

A Filter for Seasonal Adjustment

The Wiener–Kolmogorov principle is also used in deriving a filter for the seasonal adjustment of monthly and quarterly econometric data. The filter is derived from a model that combines a white-noise component $\eta(t)$ with a seasonal component obtained by passing an independent white noise $\nu(t)$ through a rational filter with poles located on the unit circle at angles corresponding to the seasonal frequencies and with corresponding zeros at the same angles but located inside the circle. The z-transform of the output sequence gives

$$\begin{aligned} y(z) &= \eta(z) + \frac{R(z)}{S(z)}\nu(z) \quad \text{or} \\ S(z)y(z) &= S(z)\eta(z) + R(z)\nu(z), \end{aligned} \tag{17}$$

where

$$R(z) = 1 + \rho z + \rho^2 z^2 + \cdots + \rho^{s-1}z^{s-1} \tag{18}$$

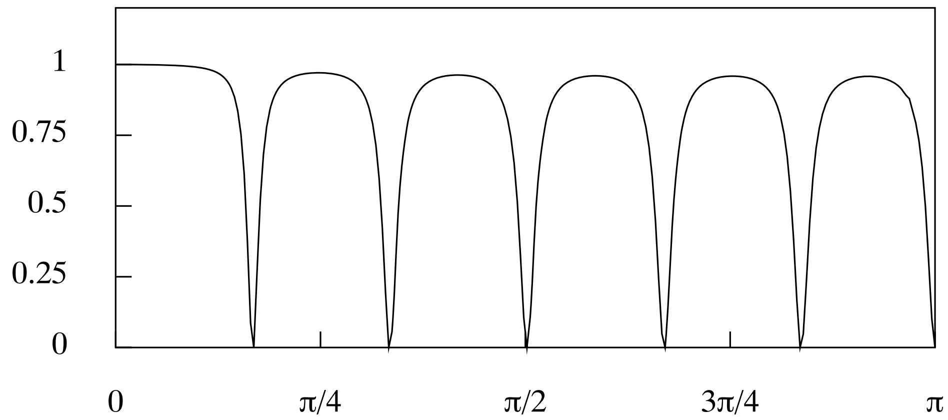

Figure 8. The gain of a filter for the seasonal adjustment of monthly data. The defining parameters are $\rho = 0.9$ and $\lambda = \sigma_\eta^2/\sigma_\nu^2 = 0.125$

with $\rho < 1$, and

$$S(z) = 1 + z + z^2 + \cdots + z^{s-1}. \tag{19}$$

The z-transform of the seasonal-adjustment filter is

$$\beta(z) = \frac{\sigma_\eta^2 S(z)S(z^{-1})}{S(z)S(z^{-1})\sigma_\eta^2 + \sigma_\nu^2 R(z)R(z^{-1})}. \tag{20}$$

Setting $z = \exp\{-\mathrm{i}\omega\}$ and letting ω run from 0 to π generates the frequency response of the filter, of which the modulus or gain is plotted in Figure 8 for the case where $\rho = 0.9$ and $\lambda = \sigma_\eta^2/\sigma_\nu^2 = 0.125$.

4. The Finite-Sample Realisations of the W–K Filters

To derive the finite-sample version of a Wiener–Kolmogorov filter, we may consider a data vector $y = [y_0, y_1, \ldots, y_{t-1},]'$ that has a signal component ξ and a noise component η:

$$y = \xi + \eta. \tag{21}$$

The two components are assumed to be independently normally distributed with zero means and with positive-definite dispersion matrices. Then,

$$\begin{aligned} E(\xi) &= 0, \qquad D(\xi) = \Omega_\xi, \\ E(\eta) &= 0, \qquad D(\eta) = \Omega_\eta, \\ \text{and} \quad & C(\xi, \eta) = 0. \end{aligned} \tag{22}$$

A consequence of the independence of ξ and η is that $D(y) = \Omega_\xi + \Omega_\eta$.

The estimates of ξ and η, which may be denoted by x and h respectively, are derived according to the following criterion:

$$\text{Minimise} \quad S(\xi,\eta) = \xi'\Omega_\xi^{-1}\xi + \eta'\Omega_\eta^{-1}\eta \quad \text{subject to} \quad \xi + \eta = y. \tag{23}$$

Since $S(\xi,\eta)$ is the exponent of the normal joint density function $N(\xi,\eta)$, the resulting estimates may be described, alternatively, as the minimum chi-square estimates or as the maximum-likelihood estimates.

Substituting for $\eta = y - \xi$ gives the concentrated criterion function $S(\xi) = \xi'\Omega_\xi^{-1}\xi + (y-\xi)'\Omega^{-1}(y-\xi)$. Differentiating this function in respect of ξ and setting the result to zero gives a condition for a minimum, which specifies the estimate x. This is $\Omega_\eta^{-1}(y-x) = \Omega_\xi^{-1}x$, which, on pre multiplication by Ω_η, can be written as $y = x - \Omega_\eta\Omega_\xi^{-1}x = (\Omega_\xi + \Omega_\eta)\Omega_\xi^{-1}x$. Therefore, the solution for x is

$$x = \Omega_\xi(\Omega_\xi + \Omega_\eta)^{-1}y. \tag{24}$$

Moreover, since the roles of ξ and η are interchangeable in this exercise, and, since $h + x = y$, there are also

$$h = \Omega_\eta(\Omega_\xi + \Omega_\eta)^{-1}y \qquad \text{and} \qquad x = y - \Omega_\eta(\Omega_\xi + \Omega_\eta)^{-1}y. \tag{25}$$

The filter matrices $B_\xi = \Omega_\xi(\Omega_\xi + \Omega_\eta)^{-1}$ and $B_\eta = \Omega_\eta(\Omega_\xi + \Omega_\eta)^{-1}$ of (24) and (25) are the matrix analogues of the z-transforms displayed in equations (4) and (5).

A simple procedure for calculating the estimates x and h begins by solving the equation

$$(\Omega_\xi + \Omega_\eta)b = y \tag{26}$$

for the value of b. Thereafter, one can generate

$$x = \Omega_\xi b \qquad \text{and} \qquad h = \Omega_\eta b. \tag{27}$$

If Ω_ξ and Ω_η correspond to the narrow-band dispersion matrices of moving-average processes, then the solution to equation (26) may be found via a Cholesky factorisation that sets $\Omega_\xi + \Omega_\eta = GG'$, where G is a lower-triangular matrix with a limited number of nonzero bands. The system $GG'b = y$ may be cast in the form of $Gp = y$ and solved for p. Then, $G'b = p$ can be solved for b. The procedure has been described by Pollock (2000).

Filters for Short Trended Sequences

To adapt these estimates to the case of trended data sequences may require the provision of carefully determined initial conditions with which to start the recursive processes. A variety of procedures are available that are similar, if not identical, in their outcomes. The procedures that are followed in the program depend upon reducing the data sequences to stationarity, in one way or another, before subjecting them to the filters. After the data have been filtered, the trend is liable to be restored.

The first method, which is the simplest in concept, requires the trend to be represented by a polynomial function. In some circumstances, when the economy has been experiencing steady growth, the polynomial will serve as a reasonable characterisation of its underlying trajectory. Thus, in the period 1955–1994 a log-linear trend function provides a firm benchmark against which to measure the cyclical fluctuations of the U.K. economy. The residual deviations from this trend may be subjected to a lowpass filter; and the filtered output can be added to the trend to produce a representation of what is commonly described as the trend-cycle component.

It is desirable that the polynomial trend should interpolate the scatter of points at either end of the data sequence. For this purpose, the program provides a method of weighted least-squares polynomial regression with a wide choice of weighting schemes, which allow extra weight to be placed upon the initial and the final runs of observations.

An alternative way of eliminating the trend is to take differences of the data. Usually, twofold differencing is appropriate. The matrix analogue of the second-order backwards difference operator in the case of $T = 5$ is given by

$$\nabla_5^2 = \begin{bmatrix} Q'_* \\ Q' \end{bmatrix} = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]. \tag{28}$$

The first two rows, which do not produce true differences, are liable to be discarded. In general, the p-fold differences of a data vector of T elements will be obtained by pre multiplying it by a matrix Q' of order $(T - p) \times T$. Applying Q' to equation (21) gives

$$\begin{aligned} Q'y &= Q'\xi + Q'\eta \\ &= \delta + \kappa = g. \end{aligned} \tag{29}$$

The dispersion matrices of the differenced vectors are

$$D(\delta) = \Omega_\delta = Q'D(\xi)Q \qquad \text{and} \qquad D(\kappa) = \Omega_\kappa = Q'D(\eta)Q. \tag{30}$$

The estimates d and k of the differenced components are given by

$$d = \Omega_\delta(\Omega_\delta + Q'\Omega_\eta Q)^{-1}Q'y \tag{31}$$

and

$$k = Q'\Omega_\eta Q(\Omega_\delta + Q'\Omega_\eta Q)^{-1}Q'y. \tag{32}$$

To obtain estimates of ξ and η, the estimates of their difference versions must be re-inflated via an anti-differencing or summation operator. We begin by observing that the inverse of ∇_5^2 is a twofold summation operator given by

$$\nabla_5^{-2} = [\, S_* \quad S\,] = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 \end{array}\right]. \tag{33}$$

The first two columns, which constitute the matrix S_*, provide a basis for all linear functions defined on $\{t = 0, 1, \ldots, T-1 = 5\}$. The example can be generalised to the case of a matrix ∇_T^{-p} of order T. However, in the program, the maximum order of differencing is $p = 2$.

We observe that, if $g_* = Q_*'y$ and $g = Q'y$ are available, then y can be recovered via the equation

$$y = S_* g_* + Sg. \tag{34}$$

In effect, the elements of g_*, which may be regarded as polynomial parameters, provide the initial conditions for the process of summation or integration, which we have been describing as a process of re-inflation.

The equations by which the estimates of ξ and η may be recovered from those of δ and κ are analogous to equation (34). They are

$$x = S_* d_* + Sd \qquad \text{and} \qquad h = S_* k_* + Sk. \tag{35}$$

In this case, the initial conditions d_* and k_* require to be estimated. The appropriate estimates are the values that minimise the function

$$\begin{aligned} (y-x)'\Omega_\eta^{-1}(y-x) &= (y - S_* d_* - Sd)'\Omega_\eta^{-1}(y - S_* d_* - Sd) \\ &= (S_* k_* + Sk)'\Omega_\eta^{-1}(S_* k_* + Sk). \end{aligned} \tag{36}$$

These values are

$$k_* = -(S_*'\Omega_\eta^{-1}S_*)^{-1}S_*'\Omega_\eta^{-1}Sk \tag{37}$$

and

$$d_* = (S_*'\Omega_\eta^{-1}S_*)^{-1}S_*'\Omega_\eta^{-1}(y - Sd). \tag{38}$$

Equations (37) and (38) together with (31) and (32) provide a complete solution to the problem of estimating the components of the data. However, it is possible to eliminate the initial conditions from the system of estimating equations. This can be achieved with the help of the following identity:

$$\begin{aligned} P_* &= S_*(S_*'\Omega_\eta^{-1}S_*)^{-1}S_*'\Omega_\eta^{-1} \\ &= I - \Omega_\eta Q(Q'\Omega_\eta Q)^{-1}Q' = I - P_Q. \end{aligned} \tag{39}$$

In these terms, the equation of (35) for h becomes $h = (I - P_*)Sk = P_Q Sk$. Using the expression for k from (32) together with the identity $Q'S = I_{T-2}$ gives

$$h = \Omega_\eta Q(\Omega_\delta + Q'\Omega_\eta Q)^{-1}Q'y. \tag{40}$$

This can also be obtained from the equation (32) for k by the removal of the leading differencing matrix Q'. It follows immediately that

$$\begin{aligned} x &= y - h \\ &= y - \Omega_\eta Q(\Omega_\delta + Q'\Omega_\eta Q)^{-1}Q'y. \end{aligned} \tag{41}$$

The elimination of the initial conditions is due to the fact that η is a stationary component. Therefore, it requires no initial conditions other than the zeros that are the appropriate estimates of the pre-sample elements. The direct estimate x of ξ does require initial conditions, but, in view of the adding-up conditions of (21), x can be obtained more readily by subtracting from y the estimate h of η, in the manner of equation (41).

Observe that, since

$$f = S_*(S_*'S_*)^{-1}S_*'y \tag{42}$$

is an expression for the vector of the ordinates of a polynomial function fitted to the data by an ordinary least-squares regression, the identity of (39) informs us that

$$f = y - Q(Q'Q)^{-1}Q'y \tag{43}$$

is an alternative expression.

The residuals of an OLS polynomial regression of degree p, which are given by $y - f = Q(Q'Q)^{-1}Q'y$, contain same the information as the vector $g = Q'y$ of the p-th differences of the data. The difference operator has the effect of nullifying the element of zero frequency and of attenuating radically the adjacent low-frequency elements. Therefore, the low-frequency spectral structures of the data are not perceptible in the periodogram of the differenced sequence. Figure 9 provides evidence of this.

On the other hand, the periodogram of a trended sequence is liable to be dominated by its low-frequency components, which will mask the other spectral structures. However, the periodogram of the residuals of the polynomial regression can be relied upon to reveal the spectral structures at all frequencies. Moreover, by varying the degree p of the polynomial, one is able to alter the relative emphasis that is given to high-frequency and low-frequency structures. Figure 10 shows that the low-frequency structure of the U.K. consumption data is fully evident in the periodogram of the residuals from fitting a linear trend to the logarithmic data.

A Flexible Smoothing Filter

A derivation of the estimator of ξ is available that completely circumvents the problem of the initial conditions. This can be illustrated with the case of a

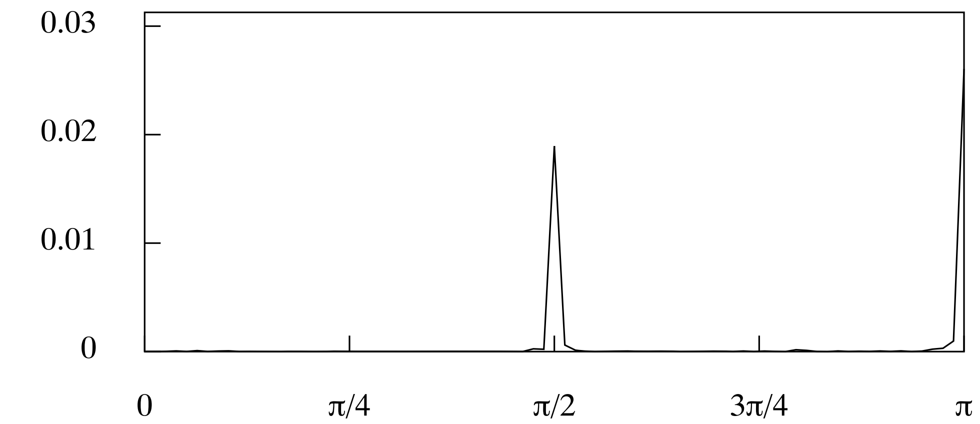

Figure 9. The periodogram of the first differences of the U.K. logarithmic consumption data.

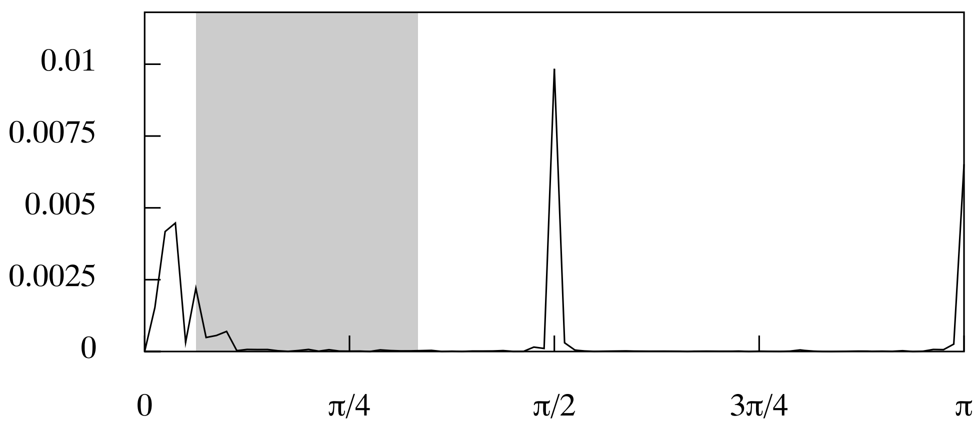

Figure 10. The periodogram of the residual sequence obtained from the linear detrending of the logarithmic consumption data. A band, with a lower bound of $\pi/16$ radians and an upper bound of $\pi/3$ radians, is masking the periodogram.

generalised version of the Leser (1961) filter in which the smoothing parameter is permitted vary over the course of the sample. The values of the smoothing parameter are contained in the diagonal matrix $\Lambda = \text{diag}\{\lambda_0, \lambda_1, \ldots, \lambda_{T-1}\}$. Then, the criterion for finding the vector is to minimise

$$L = (y - \xi)'(y - \xi) + \xi' Q \Lambda Q' \xi. \tag{44}$$

The first term in this expression penalises departures of the resulting curve from the data, whereas the second term imposes a penalty for a lack of smoothness in the curve. The second term comprises $d = Q'\xi$, which is the vector of the p-th order differences of ξ. The matrix Λ serves to generalise the overall measure of the curvature of the function that has the elements of ξ as its sampled ordinates, and it serves to regulate the penalty for roughness, which may vary over the sample.

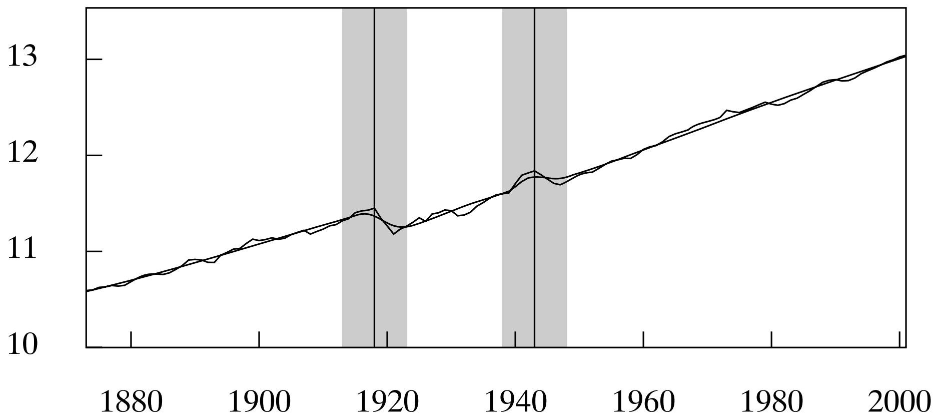

Figure 11. The logarithms of annual U.K. real GDP from 1873 to 2001 with an interpolated trend. The trend is estimated via a filter with a variable smoothing parameter.

Differentiating L with respect to ξ and setting the result to zero, in accordance with the first-order conditions for a minimum, gives

$$y - x = Q\Lambda Q'x = Q\Lambda d. \tag{45}$$

Multiplying the equation by Q' gives $Q'(y - x) = Q'y - d = Q'Q\Lambda d$, whence $\Lambda d = (\Lambda^{-1} + Q'Q)^{-1}Q'y$. Putting this into the equation $x = y - Q\Lambda d$ gives

$$\begin{aligned} x &= y - Q(\Lambda^{-1} + Q'Q)^{-1}Q'y \\ &= y - Q\Lambda(I + \Lambda Q'Q)^{-1}Q'y. \end{aligned} \tag{46}$$

This filter has been implemented in the program under the guise of a variable smoothing procedure. By giving a high value to the smoothing parameter, a stiff curve can be generated, which approaches a straight line as $\lambda \to \infty$. On the other hand, structural breaks can be accommodated by greatly reducing the value of the smoothing parameter in their neighbourhood. When $\lambda \to 0$, the filter tends to transmit the unaltered data values.

Figure 11 shown an example of the use of this filter. There were brief disruptions to the steady upwards progress of GDP in the U.K. after the two world wars. These breaks have been absorbed into the trend by reducing the value of the smoothing parameter in their localities. By contrast, the break that is evident in the data following the year 1929 has not been accommodated in the trend.

A Seasonal-Adjustment Filter

The need for initial conditions cannot be circumvented in cases where the seasonal adjustment filter is applied to trended sequences. Consider the filter that is applied to the differenced data $g = Q'y$ to produce a seasonally-adjusted sequence q. Then, there is

$$q = Q_S'(Q_S'Q_S + \lambda^{-1}Q_R'Q_R)^{-1}Q_S'g, \tag{47}$$

where Q'_R and Q'_S are the matrix counterparts of the polynomial operators $R(z)$ and $S(z)$ of (18) and (19) respectively. The seasonally adjusted version of the original trended data will be obtained by re-inflating the filtered sequence q via the equation

$$j = S_* q_* + Sq, \tag{48}$$

where

$$q_* = (S'_* S_*)^{-1} S'_* (y - Sq) \tag{49}$$

is the value that minimises the function

$$(y - j)'(y - j) = (y - S_* q_* + Sq)'(S_* q_* + Sq). \tag{50}$$

5. The Frequency-Domain Filters

Often, in the analysis economic data, we would profit from the availability of a sharp filter, with a rapid transition between the stop band and the pass band that is capable of separating components of the data that lie in closely adjacent frequency bands.

An example of the need for such a filter is provided by a monthly data sequence with an annual seasonal pattern superimposed on a trend–cycle trajectory. The fundamental seasonal frequency is of $\pi/6$ radians or 30 degrees per month, whereas the highest frequency of the trend–cycle component is liable to exceed $\pi/9$ radians or 20 degrees. This leaves a narrow frequency interval in which a filter that is intended to separate the trend–cycle component from the remaining elements must make the transition from its pass band to its stop band.

To achieve such a sharp transition, a FIR or moving-average filter requires numerous coefficients covering a wide temporal span. Such filters are inappropriate to the short data sequences that are typical of econometric analyses. Rational filters or feedback filters, as we have described them, are capable of somewhat sharper transitions, but they also have their limitations.

When a sharp transition is achieved by virtue of a rational filter with relatively many coefficients, the filter tends to be unstable on account of the proximity of some its poles to the circumference of the unit circle. (See Figure 7 for an example.) Such filters can be excessively influenced by noise contamination in the data and by the enduring effects of ill-chosen initial conditions.

A more effective way of achieving a sharp cut-off is to conduct the filtering operations in the frequency domain. Reference to equation (1) shows that an ideal filter can be obtained by replacing with zeros the Fourier coefficients that are associated with frequencies that fall within the stop band.

Complex Exponentials and the Fourier Transform

The Fourier coefficients are determined by regressing the data on the trigonometrical functions of the Fourier frequencies according to the following formulae:

$$\alpha_j = \frac{2}{T}\sum_t y_t \cos\omega_j t, \quad \text{and} \quad \beta_j = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{51}$$

Also, there is $\alpha_0 = T^{-1}\sum_t y_t = \bar{y}$, and, in the case where $T = 2n$ is an even number, there is $\alpha_n = T^{-1}\sum_t(-1)^t y_t$.

It is more convenient to work with complex Fourier coefficients and with complex exponential functions in place sines and cosines. Therefore, we define

$$\zeta_j = \frac{\alpha_j - \mathrm{i}\beta_j}{2}. \tag{52}$$

Since $\cos(\omega_j t) - \sin(\omega_j t) = e^{-\mathrm{i}\omega_j t}$, it follows that the complex Fourier transform and its inverse are given by

$$\zeta_j = \frac{1}{T}\sum_{t=0}^{T-1} y_t e^{-\mathrm{i}\omega_j t} dt \quad \longleftrightarrow y_t = \sum_{j=0}^{T-1} \zeta_j e^{\mathrm{i}\omega_j t}, \tag{53}$$

where $\zeta_{T-j} = \zeta_j^* = (\alpha_j + \beta_j)/T$. For a matrix representation of these transforms, one may define

$$\begin{aligned} U &= T^{-1/2}[\exp\{-\mathrm{i}2\pi tj/T\}; t, j = 0, \ldots, T-1], \\ \bar{U} &= T^{-1/2}[\exp\{\mathrm{i}2\pi tj/T\}; t, j = 0, \ldots, T-1], \end{aligned} \tag{54}$$

which are unitary complex matrices such that $U\bar{U} = \bar{U}U = I_T$. Then,

$$\zeta = T^{-1/2}Uy \quad \longleftrightarrow \quad y = T^{1/2}\bar{U}\zeta, \tag{55}$$

where $y = [y_0, y_1, \ldots y_{T-1}]'$ and $\zeta = [\zeta_0, \zeta_1, \ldots \zeta_{T-1}]'$ are the vectors of the data and of their spectral ordinates, respectively.

This notation can be used to advantage for representing the process of applying an ideal frequency-selective filter. Let J be a diagonal selection matrix of order T of zeros and units, wherein the units correspond to the frequencies of the pass band and the zeros to those of the stop band. Then, the selected Fourier ordinates are the nonzero elements of the vector $J\zeta$. By an application of the inverse Fourier transform, the selected elements are carried back to the time domain to form the filtered sequence. Thus, there is

$$x = \bar{U}JUy = \Psi y. \tag{56}$$

Here, $\bar{U}JU = \Psi = [\psi^{\circ}_{|i-j|}; i, j = 0, \ldots, T-1]$ is a circulant matrix of the filter coefficients that would result from wrapping the infinite sequence of the ideal bandpass

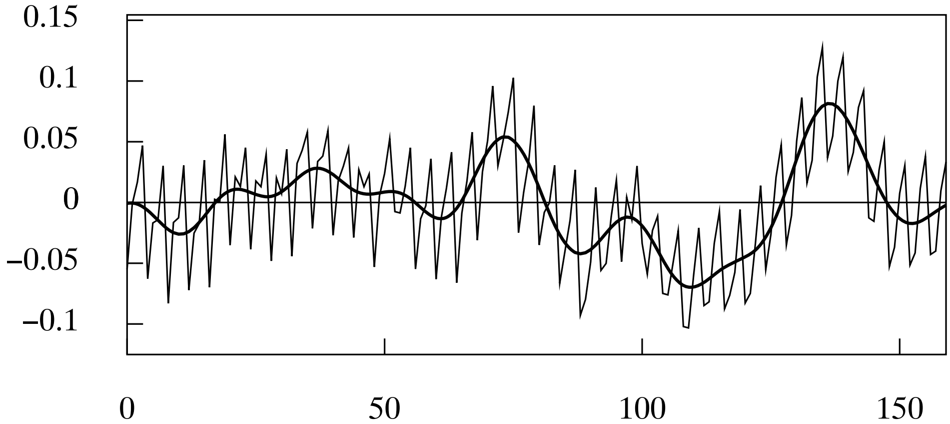

Figure 12. The residual sequence from fitting a linear trend to the logarithmic consumption data with an interpolated line representing the business cycle.

coefficients around a circle of circumference T and adding the overlying elements. Thus

$$\psi_k^\circ = \sum_{q=-\infty}^{\infty} \psi_{qT+k}. \tag{57}$$

Applying the wrapped filter to the finite data sequence via a circular convolution is equivalent to applying the original filter to an infinite periodic extension of the data sequence. In practice, the wrapped coefficients of the time-domain filter matrix Ψ would be obtained from the Fourier transform of the vector of the diagonal elements of the matrix J. However, it is more efficient to perform the filtering by operating upon the Fourier ordinates in the frequency domain, which is how the program operates.

The method of frequency-domain filtering can be used to mimic the effects of any linear-time invariant filter, operating in the time domain, that has a well-defined frequency-response function. All that is required is to replace the selection matrix J of equation (59) by a diagonal matrix containing the ordinates of the desired frequency response, sampled at points corresponding to the Fourier frequencies.

In the case of the Wiener–Kolmogorov filters, defined by equation (24) and (25), one can consider replacing the dispersion matrices Ω_ξ and Ω_η by their circular counterparts

$$\Omega_\xi^\circ = \bar{U}\Lambda_\xi U \qquad \text{and} \qquad \Omega_\eta^\circ = \bar{U}\Lambda_\eta U. \tag{58}$$

Here, Λ_ξ and Λ_η are diagonal matrices containing ordinates sampled from the spectral density functions of the respective processes. The resulting equations for the filtered sequences are

$$x = \Omega_\xi^\circ(\Omega_\xi^\circ + \Omega_\eta^\circ)^{-1}y = \bar{U}\Lambda_\xi(\Lambda_\xi + \Lambda_\eta)^{-1}Uy = \bar{U}J_\xi Uy \tag{59}$$

and

$$h = \Omega_\eta^\circ(\Omega_\xi^\circ + \Omega_\eta^\circ)^{-1}y = \bar{U}\Lambda_\eta(\Lambda_\xi + \Lambda_\eta)^{-1}Uy = \bar{U}J_\eta Uy. \tag{60}$$

An example of the application of the lowpass frequency-domain filter is provided by Figure 12. Here, a filter with a precise cut-off frequency of $\pi/8$ radians has been applied to the residuals from the linear detrending of the logarithms of the U.K. consumption data.

The appropriate cut-off frequency for this filter has been indicated by the periodogram of Figure 10. The smooth curve that has been interpolated through these residuals has been constituted from the Fourier ordinates in the interval $[0, \pi/8]$.

The same residual sequence has also been subjected to the approximate bandpass filter of Christiano and Fitzgerald (2003) to generate the estimate business cycle of Figure 4. This estimate fails to capture some of the salient low-frequency fluctuations of the data.

The highlighted region Figure 10 also show the extent of the pass band of the bandpass filter; and it appears that the low-frequency structure of the data falls mainly below this band. The fact that, nevertheless, the filter of Christiano and Fitzgerald does reflect a small proportion of the low-frequency fluctuations is due to its substantial leakage over the interval $[0, \pi/16]$, which falls within its nominal stop band.

Extrapolations and Detrending

To apply the frequency-domain filtering methods, the data must be free of trend. This can be achieved either by differencing the data or by applying the filter to data that are residuals from fitting a polynomial trend. The program has a facility for fitting a polynomial time trend of a degree not exceeding 15. To avoid the problems of collinearity that arise in fitting ordinary polynomials specified in terms of the powers of the temporal index t, a flexible generalised least-squares procedure is provided that depends upon a system of orthogonal polynomials.

In applying the methods, it is also important to ensure that there are no significant disjunctions in the periodic extension of the data at the points where the end of one replication of the sample sequence joins the beginning of the next replication. Equivalently, there must be a smooth transition between the start and finish points when the sequence of T data points is wrapped around a circle of circumference T.

The conventional means of avoiding such disjunctions is to taper the mean-adjusted, detrended data sequence so that both ends decay to zero. (See Bloomfield 1976, for example.) The disadvantage of this recourse is that it falsifies the data at the ends of the sequence, which is particularly inconvenient if, as is often the case in economics, attention is focussed on the most recent data. To avoid this difficulty, the tapering can be applied to some extrapolations, which can be added to the data, either before or after it has been detrended.

In the first case, a polynomial is fitted to the data; and tapered versions of the residual sequence that have been reflected around the endpoints of the sample are added to the extrapolated branches of the polynomial. Alternatively, if the data show strong seasonal fluctuations, then a tapered sequence based on successive

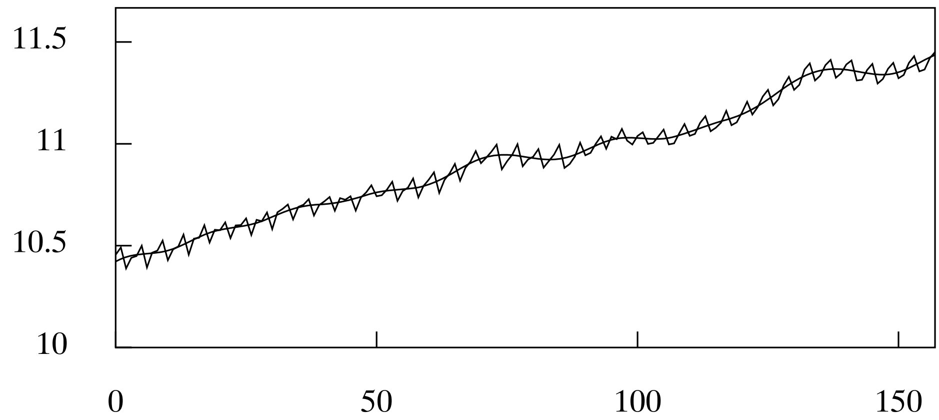

Figure 13. The trend/cycle component of U.K. Consumption determined by the frequency-domain method, superimposed on the logarithmic data.

repetitions of the ultimate seasonal cycle is added to the upper branch, and a similar sequence based on the first cycle is added to the lower branch.

In the second case, where the data have already been detrended, by the subtraction of a polynomial trend or by the application of the differencing operator, the extrapolations will be added to the horizontal axis.

This method of extrapolation will prevent the end of the sample from being joined directly to its beginning. When the data are supplemented by extrapolations, the circularity of the filter will effect only the furthest points the extrapolations, and the extrapolations will usually be discarded after the filtering has taken place. However, in many cases, extrapolations and their associated tapering will prove to be unnecessary. A case in point is provided by the filtering of the residual sequence of the logarithmic consumption data that is illustrated by Figure 12.

Anti-Differencing

After a differenced data sequence has been filtered, it will be required to reverse the effects of the differencing via a process of re-inflation. The process can be conducted in the time domain in the manner that has been indicated in section 4, where expressions have been derived for the initial conditions that must accompany the summation operations.

However, if the filtered sequence is the product of a highpass filter and if the original data have been subjected to a twofold differencing operation, then an alternative method of re-inflation is available that operates in the frequency domain. This method is used in the program only if the filtering itself has taken place in the frequency domain.

In that case, the reduction to stationarity will be by virtue of a centralised twofold differencing operator of the form

$$(1 - z^{-1})(1 - z) = -z\nabla^2(z) \tag{61}$$

The frequency-response function of the operator, which is obtained by setting $z = \exp\{-i\omega\}$ in this equation, is

$$f(\omega) = 2 - 2\cos(\omega). \tag{62}$$

The frequency response of the anti-differencing operator is $v(\omega) = 1/f(\omega)$.

The matrix version of the centralised operator can be illustrated by the case where $T = 5$:

$$N_5 = \begin{bmatrix} n_0' \\ -Q' \\ n_4' \end{bmatrix} = -\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ \hline 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{63}$$

In applying this operator to the data, the first and the last elements of $N_T y$, which are denoted by $n_0'y$ and $n_{T-1}'y$, respectively, are not true differences. Therefore, they are discarded to leave $-Q'y = [q_1, \ldots, q_{T-2}]'$. To compensate for this loss, appropriate values are attributed to q_0 and q_{T-1}, which are formed from combinations of the adjacent values, to create a vector of order T denoted by $q = [q_0, q_1, \ldots, q_{T-2}, q_{T-1}]'$.

The highpass filtering of the data comprises the following steps. First, the vector q is translated to the frequency domain to give $\gamma = Uq$. Then, the frequency-response matrix J_η is applied to the resulting Fourier ordinates. Next, in order to compensate for the effects of differencing, the vector of Fourier ordinates is premultiplied by a diagonal matrix $V = \text{diag}\{v_0, v_1, \ldots, v_{T-1}\}$, wherein $v_j = 1/f(\omega_j); j = 0, \ldots, T-1$, with $\omega_j = 2\pi j/T$. Finally, the result is translated back to the time domain to create the vector h.

The vector of the complementary component is $x = y - h$. Thus there are

$$h = \bar{U}H_\eta Uq \qquad \text{and} \qquad x = y - \bar{U}H_\eta Uq, \tag{64}$$

where $H\eta = VJ_\eta$. It should be noted that the technique of re-inflating the data within the frequency domain cannot be applied in the case of a lowpass component for the reason that $f(0) = 0$ and, therefore, the function $v(\omega) = 1/f(\omega)$ is unbounded at the zero frequency $\omega = 0$. However, as the above equations indicate, this is no impediment to the estimation of the corresponding component x.

An example of the application of these procedures is provided by Figure 13, which concerns the familiar logarithmic consumption data, through which a smooth trend-cycle function has been interpolated. This is indistinguishable from the function that is obtained by adding the smooth business-cycle of Figure 12 to the linear trend that was subtracted from the data in the process of detrending it. The program also allows the trend-cycle function to be constructed in this manner.

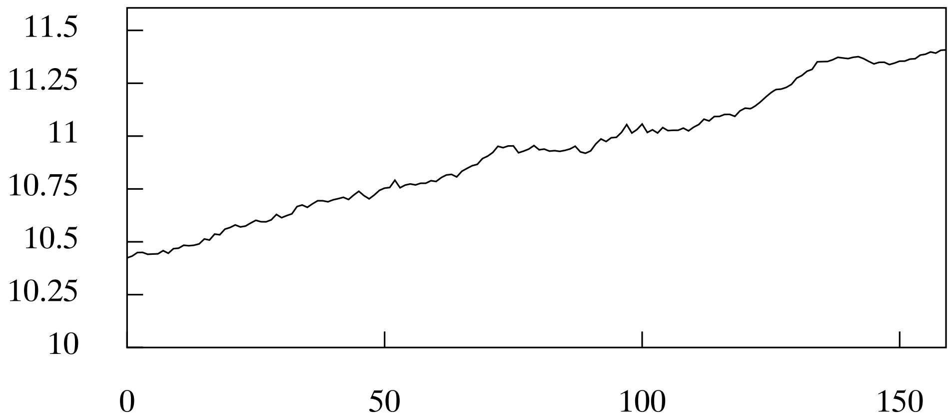

Figure 14. The plot of a seasonally adjusted version of the consumption data of Figures 2 and 13, obtained via the time domain filter

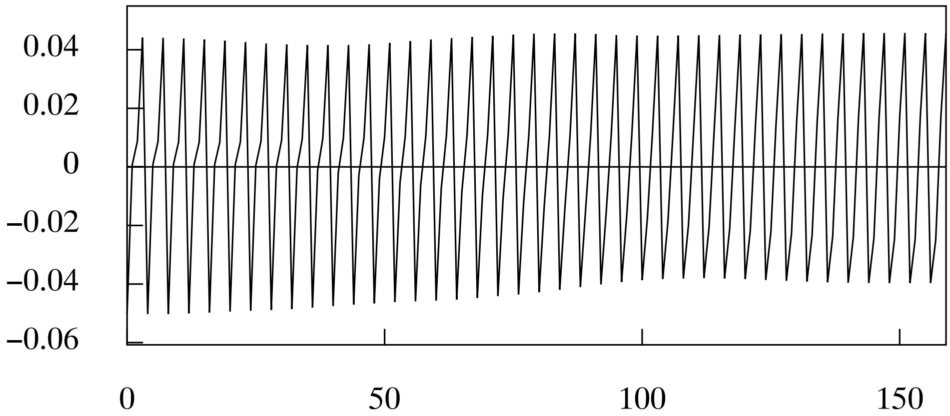

Figure 15. The seasonal component extracted from the U.K. consumption data by a time-domain filter.

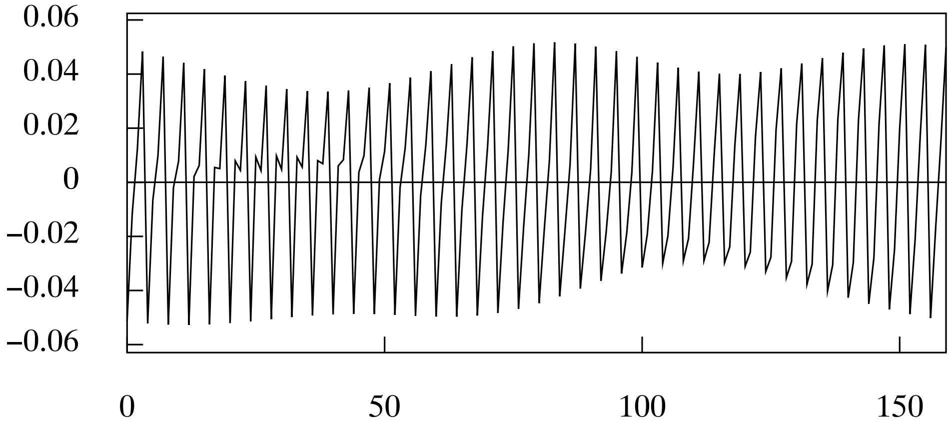

Figure 16. The seasonal component extracted from the U.K. consumption data by a frequency-domain filter.

Seasonal Adjustment in the Frequency Domain

The method of frequency-domain filtering is particularly effective in connection with the seasonal adjustment of monthly or quarterly data. It enables one to remove elements not only at the seasonal frequencies but also at adjacent frequencies by allowing one to define a neighbourhood for each of the stop bands surrounding the fundamental seasonal frequency and its harmonics.

If only the fundamental seasonal element and its harmonics are entailed in its synthesis, then the estimated seasonal component will be invariant from year to year. If elements at the adjacent frequencies are also present in the synthesis, then it will evolve gradually over the length of the sample period.

The effects of the seasonal-adjustment filters of the program are illustrated in Figures 14–16. Figure 14 shows the seasonally adjusted version of the logarithmic consumption data that has been obtained via the Wiener–Kolmogorov filter of section 4. Figure 15 shows the seasonal component that has been extracted in the process.

The regularity of this component is, to some extent, the product of the filter. Figure 16 shows a less regular seasonal component that has been extracted by the frequency-domain filter described in the present section. This component has been synthesised from elements at the Fourier frequencies and from those adjacent to them that have some prominence if the periodogram of Figure 10.

6. The Program and its Code

The code of the program that has been described in this paper is freely available at the web address that has been given. This code is in Pascal. A parallel code in C has been generated with the help of a Pascal-to-C translator, which has been written by the author. The aim has been to make the program platform-independent and to enable parts of it to be realised in other environments.

This objective has dictated some of the features of the user interface of the program, which, in its present form, eschews such devices as pull-down menus and dialogue boxes etc. Subsequent versions of the program will make limited use of such enhancements.

However, the nostrum that a modern computer program should have a modeless interface will be resisted. Whereas such an interface is necessary for programs such as word processors, where all of the functions should be accessible at all times, it is less appropriate to statistical programs where, in most circumstances, the user will face a restricted set of options. Indeed, the present program is designed to restrict the options, at each stage of the operations, to those that are relevant.

A consequence of this design is that there is no need of a manual of instructions to accompany the program. Instead, the three log files that record the steps taken in filtering some typical data sequences should provide enough help to get the user underway. What is more important is that the user should understand the nature of the statistical procedures that have been implemented; and this has been the purpose of the present paper.

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