The 2nd-order Difference Equation with Complex Roots

Consider the 2nd-order equation

$$\alpha_0 y(t) + \alpha_1 y(t-1) + \alpha_2 y(t-2) = 0, \tag{1}$$

and imagine that the auxilliary equation

$$\alpha_0 z^2 + \alpha_1 z + \alpha_2 = 0 \tag{2}$$

has complex roots μ and μ_*. These can be written as

$$\begin{aligned} \mu &= \gamma + i\delta = \kappa(\cos\omega + i\sin\omega) = \kappa e^{i\omega}, \\ \mu_* &= \gamma - i\delta = \kappa(\cos\omega - i\sin\omega) = \kappa e^{-i\omega}. \end{aligned} \tag{3}$$

where $\kappa = \sqrt{\gamma^2 + \delta^2}$ and $\omega = \tan^{-1}(\delta/\gamma)$. The general solution of the difference equation is given by

$$y(t) = c\mu^t + c_*(\mu_*)^t. \tag{4}$$

This is a real-valued sequence, and, since a real term must equal its own conjugate, we require c and c_* to be conjugate numbers of the form

$$\begin{aligned} c_* &= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ c &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}. \end{aligned} \tag{5}$$

Thus we have

$$\begin{aligned} y(t) = c\mu^t + c_*(\mu_*)^t &= \rho e^{-i\theta}(\kappa e^{i\omega})^t + \rho e^{i\theta}(\kappa e^{-i\omega})^t \\ &= \rho\kappa^t \left\{ e^{i(\omega t - \theta)} + e^{-i(\omega t - \theta)} \right\} \\ &= 2\rho\kappa^t \cos(\omega t - \theta). \end{aligned} \tag{6}$$

It is convenient to write this in the form of

$$y(t) = \kappa^t \alpha \cos(\omega t) + \kappa^t \beta \sin(\omega t),$$

where $\alpha = 2\rho\cos\theta$ and $\beta = 2\rho\sin\theta$. This comes from using the identity $\cos(\omega t - \theta) = \cos\theta\cos(\omega t) + \sin\theta\sin(\omega t)$

Let $\alpha_1 = -1.2$ and $\alpha_2 = 0.72$ in equation (1). Then

$$\begin{aligned} \gamma \pm i\delta &= 0.6 \pm 0.6 \\ \omega &= \frac{\pi}{4}, \quad \text{and} \\ \kappa &= \sqrt{0.72}. \end{aligned} \tag{7}$$

With $y_0 = 4$ and $y_1 = 3$ we have

$$\begin{aligned} \kappa^0\{\alpha\cos 0 + \beta\sin 0\} &= 4 \quad \text{and} \\ \kappa\left\{\alpha\cos\frac{\pi}{4} + \beta\sin\frac{\pi}{4}\right\} &= 3. \end{aligned} \tag{8}$$

Since $\cos 0 = 1$ and $\sin 0 = 0$, the first equation yields $\alpha = 4$. Since $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, the second equation becomes

$$\frac{1}{\sqrt{2}}\{\alpha + \beta\} = \frac{3}{\kappa}, \tag{9}$$

which yields $\beta = (3\sqrt{2}/\sqrt{0.72}) - 4 = 1$.