The Autocovariances of an AR(2) Process

The Yule–Walker equations of an AR(2) process can be written in two ways:

$$(1)\qquad \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_2 & \alpha_1 & 1 & 0 & 0 \\ 0 & \alpha_2 & \alpha_1 & 1 & 0 \\ 0 & 0 & \alpha_2 & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_2 \\ \gamma_1 \\ \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1+\alpha_2 & 0 \\ \alpha_2 & \alpha_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

If we know the values for $\gamma_0, \gamma_1, \gamma_2$, then we can solve the equations to give the parameter values

$$(2)\qquad \alpha_1 = \frac{\gamma_1\gamma_2 - \gamma_0\gamma_1}{\gamma_0^2 - \gamma_1^2}, \qquad \alpha_2 = \frac{\gamma_1^2 - \gamma_0\gamma_2}{\gamma_0^2 - \gamma_1^2}.$$

Then we can find $\sigma_\varepsilon^2 = \gamma_0 + \alpha_1\gamma_1 + \alpha_2\gamma_2$. Conversely, to obtain the first two autocovariances, we reduce the equation to

$$(3)\qquad \begin{bmatrix} 1-\alpha_2^2 & \alpha_1(1-\alpha_2) \\ \alpha_1 & 1+\alpha_2 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \end{bmatrix}.$$

Solving this gives

$$(4)\qquad \gamma_0 = \frac{\sigma_\varepsilon^2(1+\alpha_2)}{(1-\alpha_2)(1+\alpha_2+\alpha_1)(1+\alpha_2-\alpha_1)}, \qquad \gamma_1 = \frac{-\sigma_\varepsilon^2\alpha_1}{(1-\alpha_2)(1+\alpha_2+\alpha_1)(1+\alpha_2-\alpha_1)}.$$

Given the values of γ_0 and γ_1, we can proceed to find $\gamma_2 = -\alpha_2\gamma_0 - \alpha_1\gamma_1$.

When $\alpha_1 = 1$, $\alpha_2 = 0.5$ and $\sigma^2\varepsilon = 1$, we find that $\gamma_0 = 12/5$,$\gamma_1 = -8/5$ and $\gamma_2 = -2/5$.