BURMAN'S METHOD OF SIGNAL EXTRACTION: THE START-UP PROBLEM

Consider a seasonal ARIMA model of the form

$$y(t) = \frac{\theta(L)}{\psi(L)}\varepsilon(t) = \frac{\theta(L)}{\psi_S(L)\psi_T(L)}\varepsilon(t), \tag{1}$$

where $\psi_S(L)$ stands for the seasonal factors and $\psi_T(L)$ stands for the trend factors. We assume that there is a sample of T observations on $y(t)$, running from $t = 0$ to $t = T - 1$.

The autocovaraince generating function of the model can be decomposed into three components, which correspond to the trend effect, the seasonal effect and an irregular influence. Thus

$$\frac{\theta(z)\theta(z^{-1})}{\psi_S(z)\psi_T(z)\psi_T(z^{-1})\psi_S(z^{-1})} = \frac{Q_T(z)}{\psi_T(z)\psi_T(z^{-1})} + \frac{Q_S(z)}{\psi_S(z)\psi_S(z^{-1})} + R(z). \tag{2}$$

According to Wiener–Kolmogorov theory, the optimal signal-extraction filter for the seasonal component is

$$\begin{aligned}\beta_S(L) &= \frac{Q_S(L)}{\psi_S(L)\psi_S(L^{-1})} \times \frac{\psi_S(L)\psi_T(L)\psi_T(L^{-1})\psi_S(L^{-1})}{\theta(L)\theta(L^{-1})}\\ &= \frac{Q_S(L)\psi_T(L)\psi_T(L^{-1})}{\theta(L)\theta(L^{-1})} = \frac{D_S(L)}{\theta(L)\theta(L^{-1})}.\end{aligned} \tag{3}$$

Using a partial-fraction decomposition, this can be written as

$$\frac{D_S(L)}{\theta(L)\theta(L^{-1})} = \frac{G_S(L)}{\theta(L)} + \frac{G_S(L^{-1})}{\theta(L^{-1})}. \tag{4}$$

The estimate of the seasonal component is therefore

$$s(t) = f(t) + b(t) = \frac{G_S(L)}{\theta(L)}y(t) + \frac{G_S(L^{-1})}{\theta(L^{-1})}y(t). \tag{5}$$

This consists of a component $f(t)$, obtained by running forwards through the data, and a component $b(t)$, obtained by running backwards through the data.

In order to compute either of these components, one needs some initial conditions. Consider the recursion running backward through the data, which is associated with the equation

$$\theta(L^{-1})b(t) = G_S(L^{-1})y(t). \tag{6}$$

This requires some post-sample values of both $b(t)$ and $y(t)$ for its initial conditions. The post-sample values of $y(t)$ are generated in the usual way using a recursion based upon the equation of the ARMA model, which is

$$\psi(L)y(t) = \theta(L)\varepsilon(t). \tag{7}$$

Here the requisite post-sample elements of $\varepsilon(t)$ are represented by their zero-valued expectations.

Next, in order to find the post-sample values of $b(t)$, we need to consider the equation

$$\psi(L)b(t) = \psi(L)\frac{G_S(L^{-1})}{\theta(L^{-1})}y(t) = \frac{G_S(L^{-1})}{\theta(L^{-1})}\psi(L)y(t) = v(t). \tag{8}$$

Given that the post-sample elements of $\varepsilon(t)$ for $t > T-1$ are represented by zeros, it follows that the elements of $\psi(L)y(t) = \theta(L)\varepsilon(t)$ are zero-valued for $t \geq T+q$, where q is the degree of the polynomial $\theta(L)$. The elements of $v(t)$, which are combinations of the elements of $\psi(L)y(t)$, are likewise zero-valued for $t \geq T+q$.

There are therefore two sets of equations, obtained from (6) and (8), respectively, from which the post-sample values of $b(t)$ can be deduced; and these can be written as

$$\begin{aligned} &\theta(L^{-1})b(t) = w(t), \quad \text{where} \quad w(t) = G_S(L^{-1})y(t), \quad \text{and} \\ &\psi(L)b(t) = v(t), \quad \text{where} \quad v(t) = 0 \quad \text{for} \quad t > T+q. \end{aligned} \tag{9}$$

We can proceed as follows. First the post-sample values of $y(t)$ are forecast for $t = T, \ldots, T+q+r-1$, where $r = \max(p, q)$ is the maximum of the orders autoregressive and moving average operators in equation (7). Next, the values of the intermediate series $w(t) = G_S(L^{-1})y(t)$ are found for $t = 0, \ldots, T+q-1$. The latter are sufficient to enable us to form the equation $\theta(L^{-1})b(t) = w(t)$ of (9) for $t = T+q-p, \ldots, T+q-1$. We can also form the equations $\psi(L)b(t) = 0$ for the ensuing dates $t = T+q, \ldots, T+2q-1$. The two sets of equations can be combined to form the following system:

$$\left[\begin{array}{ccccccc} \theta_0 & \theta_1 & \ldots & \theta_q & 0 & \ldots & 0 \\ 0 & \theta_0 & \ldots & \theta_{q-1} & \theta_q & \ldots & 0 \\ \vdots & \vdots & \ddots & & & \ddots & \vdots \\ 0 & 0 & \ldots & \theta_0 & \theta_1 & \ldots & \theta_q \\ \hline \psi_p & \psi_{p-1} & \ldots & \psi_0 & 0 & \ldots & 0 \\ 0 & \psi_p & \ldots & \psi_1 & \psi_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & & \ddots & \vdots \\ 0 & 0 & \ldots & \psi_p & \psi_{p-1} & \ldots & \psi_0 \end{array}\right] \begin{bmatrix} b_{T+q-p} \\ b_{T+q-p+1} \\ \vdots \\ b_{T+q-1} \\ b_{T+q} \\ b_{T+q+1} \\ \vdots \\ b_{T+2q-1} \end{bmatrix} = \begin{bmatrix} w_{T+q-p} \\ w_{T+q-p+1} \\ \vdots \\ w_{T+q-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

These equations are solved to obtain the start-up values for running the recursion of (6).

A similar method can be used for finding the start-up values for running a recursion based on the equation

$$\theta(L)f(t) = G_S(L)y(t) \tag{11}$$

which is the mirror image of (6). The estimate of the seasonal component is found by adding the corresponding elements of $f(t)$ and $b(t)$.