BIVARIATE SPECTRAL ANALYSIS

Let $x(t)$ and $y(t)$ be two stationary stochastic processes with $E\{x(t)\} = E\{y(t)\} = 0$. These processes have the following spectral representations:

$$\text{(1)} \qquad \begin{aligned} x(t) &= \int_0^{\pi} \{\cos(\omega t) dA_x(\omega) + \sin(\omega t) dB_x(\omega)\}, \\ y(t) &= \int_0^{\pi} \{\cos(\omega t) dA_y(\omega) + \sin(\omega t) dB_y(\omega)\}. \end{aligned}$$

The weighting functions $A_j(\omega), B_j(\omega); j = x, y$ are a pair of mutually independent stochastic processes defined over the interval $[0, \pi]$. Their realisations correspond to the realisations of the temporal processes $x(t)$ and $y(t)$ to which they pertain.

Several conditions must be fulfilled by $A_j(\omega), B_j(\omega)$ to ensure that $x(t)$ and $y(t)$ are stationary and that their autocovariances are invariant over time. A further set of conditions must be fulfilled to ensure that the cross-covariances between the two processes are time-invariant. We shall begin with the assumptions that are internal to the two processes. Then we shall examine the assumptions that relate one process to the other.

The Assumptions Internal to the Processes

It is assumed that the functions $dA_j(\omega)$ and $dB_j(\omega)$ represent a pair of stochastic processes of zero mean that are indexed on the continuous frequency parameter $\omega \in [0, \pi]$. Thus

$$\text{(2)} \qquad E\big\{dA_j(\omega)\big\} = E\big\{dB_j(\omega)\big\} = 0.$$

Next, it is assumed that $A_j(\omega)$ and $B_j(\omega)$ are mutually uncorrelated and that non-overlapping increments within each process are uncorrelated. Thus

$$\text{(3)} \qquad \begin{aligned} E\big\{dA_j(\omega) dB_j(\lambda)\big\} &= 0 \quad \text{for all} \quad \omega, \lambda, \\ E\big\{dA_j(\omega) dA_j(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\big\{dB_j(\omega) dB_j(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned}$$

The final assumption affecting the individual processes $x(t)$ or $y(t)$ is that the variance of the increments is given by

$$\text{(4)} \qquad \begin{aligned} V\big\{dA_j(\omega)\big\} = V\big\{dB_j(\omega)\big\} &= 2dF_j(\omega) \\ &= 2f_j(\omega) d\omega. \end{aligned}$$

Here, $F_j(\omega)$ is a continuous differentiable function, in contrast to $A_j(\omega)$ and $B_j(\omega)$, which are continuous almost everywhere but which are nowhere differentiable. The function $F_j(\omega)$ and its derivative $f_j(\omega)$ are the spectral distribution function and the spectral density function, respectively.

To understand the effect of these various assumptions, one can begin with (4), which indicates that the increments of $A_j(\omega)$ and $B_j(\omega)$ have the same variance. The effect of the assumption is that the phase angle of the trigonometrical function at frequency ω, which is formed from the weighed combination of the sine and the cosine, is distributed uniformly over the interval $[-0, \pi]$. It would be difficult to justify a different assumption.

Next are the assumptions concerning the autocovariances of the processes. Consider

$$\begin{aligned} E(y_t y_s) = \int_0^\pi \int_0^\pi \big[&\cos(\omega t)\cos(\lambda s)E\{dA_y(\omega)dA_y(\lambda)\} \\ &+ \cos(\omega t)\sin(\lambda s)E\{dA_y(\omega)dB_y(\lambda)\} \\ &+ \sin(\omega t)\cos(\lambda s)E\{dB_y(\omega)dA_y(\lambda)\} \\ &+ \sin(\omega t)\sin(\lambda s)E\{dB_y(\omega)dB_y(\lambda)\}\big]. \end{aligned} \tag{5}$$

Given the assumptions of (3) concerning the absence of correlations amongst non-overlapping increments of $A_y(\omega)$ and $B_y(\omega)$ and the absence of correlation between the two processes, this becomes

$$\begin{aligned} E(y_t y_s) &= 2\int_0^\pi \big\{ \cos(\omega t)\cos(\omega s)f_y(\omega) + \sin(\omega t)\sin(\omega s)f_y(\omega)\big\}d\omega \\ &= 2\int_0^\pi \cos(\omega[t-s])f_y(\omega)d\omega, \end{aligned} \tag{6}$$

which follows in view of the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$. Thus, the autocovariance $E(y_t y_s) = \gamma^{yy}_{|t-s|}$ is a function of the temporal separation of the elements y_t, y_s that is independent of their absolute dates. This is a necessary condition for the stationarity of the process $y(t)$.

Equation (6) can expressed more elegantly in terms of complex exponentials. Let the domain of the $f_y(\omega)$ be extended over the negative frequencies such that $f_y(-\omega) = f_y(\omega)$. Then, using $\cos(\omega\tau) = \{\exp(\mathrm{i}\omega\tau) + \exp(-\mathrm{i}\omega\tau)\}/2$, where $\tau = |s - t|$, and denoting the autocovariance by $\gamma^{yy}_\tau = E(y_t y_s)$, equation (6) can be rendered as

$$\gamma^{yy}_\tau = \int_{-\pi}^{\pi} f_y(\omega)e^{\mathrm{i}\omega\tau}d\omega. \tag{7}$$

The inverse mapping from the autocovariances to the spectrum is given by

$$\begin{aligned} f_y(\omega) &= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\gamma_\tau^{yy}e^{-i\omega\tau} \\ &= \frac{1}{2\pi}\Big\{\gamma_0^{yy} + 2\sum_{\tau=1}^{\infty}\gamma_\tau^{yy}\cos(\omega\tau)\Big\}. \end{aligned} \tag{8}$$

This becomes a cosine Fourier transform in consequence of the symmetry of the autocovariance function whereby $\gamma_\tau^{yy} = \gamma_{-\tau}^{yy}$.

The essential interpretation of the spectral density function is indicated by the equation

$$\gamma_0^{yy} = \int_{-\pi}^{\pi} f_y(\omega)d\omega, \tag{9}$$

which comes from setting $\tau = 0$ in equation (7). This equation shows the measure in which the variance or 'power' of $y(t)$, which is γ_0^{yy}, is attributed by the spectral density function to the various cyclical components of which the process is composed.

The Assumptions Connecting the Processes

In order to determine the relatedness of the two processes $x(t), y(t)$, some assumptions are needed regarding the covariances across the processes $A_x(\omega)$, $A_y(\omega)$ and $B_x(\omega)$, $B_y(\omega)$. First, there is the assumption that there are no correlations across the frequencies:

$$\begin{aligned} &E\{dA_x(\omega)dA_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda, \\ &E\{dA_x(\omega)dB_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda, \\ &E\{dB_x(\omega)dB_y(\lambda)\} = 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned} \tag{10}$$

Next, there are two covariance relationships:

$$\begin{aligned} E\{dA_x(\omega)dA_y(\omega)\} = E\{dB_x(\omega)dB_y(\omega)\} &= 2dC_{xy}(\omega) \\ &= 2c_{xy}(\omega)d\omega \end{aligned} \tag{11}$$

and

$$\begin{aligned} E\{dA_x(\omega)dB_y(\omega)\} = -E\{dB_x(\omega)dA_y(\omega)\} &= 2dQ(\omega) \\ &= 2q_{xy}(\omega)d\omega. \end{aligned} \tag{12}$$

The first of these functions is co-spectrum $c_{xy}(\omega)$, which defines the covariances of the amplitudes of cosine components of $x(t)$ and $y(t)$ that are in phase at the frequency ω. The second is the quadrature spectrum $q_{xy}(\omega)$, which defines the covariance of the amplitudes of the sine and the cosine components of $x(t)$ and $y(t)$, which are in quadrature at the frequency ω, which is to say that they are out of phase to the extent of $\pi/2$ radians.

The effects of these assumptions may be examined in connection with the spectral representation of the covariances of $x(t)$ and $y(t)$. From the spectral representations of (1), the following quadratic product is derived:

$$
\begin{aligned}
E(x_t y_s) = \int_0^\pi \int_0^\pi \big[& \cos(\omega t)\cos(\lambda s) E\{dA_x(\omega)dA_y(\lambda)\} \\
& + \cos(\omega t)\sin(\lambda s) E\{dA_x(\omega)dB_y(\lambda)\} \\
& + \sin(\omega t)\cos(\lambda s) E\{dB_x(\omega)dA_y(\lambda)\} \\
& + \sin(\omega t)\sin(\lambda s) E\{dB_x(\omega)dB_y(\lambda)\} \big].
\end{aligned} \tag{13}
$$

However, according to (10), the random increments for the frequency ω in one process are uncorrelated with the random increments for the frequency λ in the other process. Therefore, the double integral collapses to an single integral in respect of one frequency, and, from (11) and (12), it follows that

$$
\begin{aligned}
E(x_t y_s) = \int_0^\pi \big[& \cos(\omega t)\cos(\omega s) dC_{xy}(\omega) \\
& + \cos(\omega t)\sin(\omega s) dQ_{xy}(\omega) \\
& + \sin(\omega t)\cos(\omega s) dQ_{xy}(\omega) \\
& + \sin(\omega t)\sin(\omega s) dC_{xy}(\omega) \big].
\end{aligned} \tag{14}
$$

Finally, by virtue of two trigonometrical identities, which are the cosine identities quoted above, and the analogous sine identity $\sin(A - B) = \sin A\cos B - \cos A\sin B$, we find that

$$
E(x_t y_s) = 2\int_0^\pi \big\{ \cos(\omega[t-s]) dC_{xy}(\omega) + \sin(\omega[t-s]) dQ_{xy}(\omega) \big\}. \tag{15}
$$

Let us set $dC_{xy}(\omega) = c_{xy}(\omega)d\omega$ and $dQ_{xy}(\omega) = q_{xy}(\omega)d\omega$ in accordance with the assumption that the spectral functions are differentiable, which will be true in the absence of perfectly regular periodic components in the processes $x(t)$ and $y(t)$. Then, on setting $t - s = \tau$ and using the notation $\gamma_\tau^{xy} = E(x_t, y_{t-\tau})$, equation (15) can be rewritten as

$$
\gamma_\tau^{xy} = 2\int_0^\pi \big\{ \cos(\omega\tau) c_{xy}(\omega) d\omega + \sin(\omega\tau) q_{xy}(\omega) \big\} d\omega. \tag{16}
$$

In order to express (16) in terms of complex exponentials, the so-called cross-spectrum is defined:

$$
\begin{aligned}
g^{xy}(\omega) &= c_{xy}(\omega) + \mathrm{i} q_{xy}(\omega) \\
&= \{c_{xy}^2(\omega) + q_{xy}^2(\omega)\}^{1/2} e^{\theta(\omega)},
\end{aligned} \tag{17}
$$

where $\theta(\omega) = \tan^{-1}\{q_{xy}(\omega)/c_{xy}(\omega)\}$. Also, the domain of the integration is extended from $[0, \pi]$ to $(-\pi, \pi]$ by regarding $c_{xy}(\omega)$ as an even function such that $c_{xy}(\omega) = c_{xy}(-\omega)$ and by regarding $q_{xy}(\omega)$ as an odd function such that $q_{xy}(\omega) = -q_{xy}(-\omega)$. Then, since $\cos(\omega t) = \{\exp(\mathrm{i}\omega t) + \exp(-\mathrm{i}\omega t)\}/2$ and $\sin(\omega t) = -\mathrm{i}\{\exp(\mathrm{i}\omega t) - \exp(-\mathrm{i}\omega t)\}/2$, there is

$$\gamma_\tau^{xy} = \int_{-\pi}^{\pi} g^{xy}(\omega)e^{\mathrm{i}\omega\tau}d\omega. \tag{18}$$

The inverse of this relationship indicates that the cross spectrum is the Fourier transform of the covariances of $x(t)$ and $y(t)$ in the same way that the spectral density function of $y(t)$, which is to be found under (8), is the Fourier transform of the sequence of its autocovariances:

$$g^{xy}(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau^{xy} e^{-i\omega\tau}. \tag{19}$$

Notice, however, that since, in general, $\gamma_\tau^{xy} \neq \gamma_{-\tau}^{xy}$, this does not entail a cosine Fourier transform as does the corresponding definition of the spectrum as the transform of the autocovariance function.

The even function $c_{xy}(\omega)$ is the cosine portion of $g^{xy}(\omega)$ and the odd function $q(\omega)$ is its sine portion. These quantities are defined separately by

$$c_{xy}(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \frac{\gamma_\tau^{xy} + \gamma_{-\tau}^{xy}}{2} e^{-\mathrm{i}\omega\tau} = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau^{xy}\cos(\omega\tau), \tag{20}$$

$$\mathrm{i}q_{xy}(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \frac{\gamma_\tau^{xy} - \gamma_{-\tau}^{xy}}{2} e^{-\mathrm{i}\omega\tau} = \frac{-\mathrm{i}}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau^{xy}\sin(\omega\tau), \tag{21}$$

where $(\gamma_\tau^{xy} + \gamma_{-\tau}^{xy})/2$ and $\cos(\omega\tau)$ are even or symmetric functions and $(\gamma_\tau^{xy} - \gamma_{-\tau}^{xy})/2$ and $\sin(\omega\tau)$ are odd or anti-symmetric functions. It can be confirmed that

$$\begin{aligned} &\text{(i)}\quad c_{xy}(\omega) = c_{xy}(-\omega), \qquad \text{and that}\\ &\text{(ii)}\quad q_{xy}(\omega) = -q_{xy}(-\omega) \quad \text{and} \quad \text{(iii)} \quad q_{xy}(\omega) = q_{yx}(-\omega). \end{aligned} \tag{22}$$

An insight into the nature of the cross spectrum is indicated by the equation

$$\gamma_0^{xy} = \int_{-\pi}^{\pi} g_{xy}(\omega)d\omega = \int_{-\pi}^{\pi} c_{xy}(\omega)d\omega, \tag{23}$$

which comes from setting $\tau = 0$ in equation (18) and from observing that the integral of the odd functin $q_{xy}(\omega)$ is zero. This equation shows the ordinary contemporaneous covariance between the processes $x(t)$ and $y(t)$ relates only to the cyclical components that are in phase. The concept of spectral coherence, which is expounded in the next section, also accommodates relationships between components that are out of phase.

Measures of Relatedness between Two Series

The relatedness of two stationary stochastic processes can be measured by their spectral coherence. The coherence function, which is defined over the interval $[0, \pi]$, gives the correlation of the cyclical components of the sequences at each frequency. The coherence of $x(t)$ and $y(t)$ at the frequency ω is defined by

$$(24) \qquad \rho^{xy}(\omega) = \frac{|g^{xy}(\omega)|}{\{f^{xx}(\omega)f^{yy}(\omega)\}^{1/2}} = \left\{ \frac{c^2_{xy}(\omega) + q^2_{xy}(\omega)}{f^{xx}(\omega)f^{yy}(\omega)} \right\}^{1/2}.$$

Since it takes account of both the co-spectrum and the quadrature spectrum, the measure is unaffected by the relative phase alignment of the two components.

One should recall, in this connection, that the ordinary correlation coefficient of two sinusoids of the same frequency that are in quadrature would be zero. The familiar example of a pair of trigonometric functions that are in quadrature, i.e. that are separated by a phase displacement of $\pi/2$ radians or 90°, are $\cos(\omega t)$ and $\sin(\omega t)$, which are well known to be mutually orthogonal.

The coherence at any frequency ω is, in effect, the ordinary measure of correlation which would be obtained by bringing the components of the two sequences into phase alignment. Consider the components

$$(25) \qquad \begin{aligned} x(\omega, t) &= \cos(\omega t)dA_x(\omega) + \sin(\omega t)dB_x(\omega), \\ y(\omega, t) &= \cos(\omega t - \theta)dA_y(\omega) + \sin(\omega t - \theta)dB_y(\omega), \end{aligned}$$

which are extracted from the spectral representation of the two sequences specified by (1). The second component has been translated through a phase angle of θ which we intend to adjust so as to maximise the covariance of the two component. By the algebra that has given rise to equation (15), it is found that the covariance of these components is

$$(26) \qquad \frac{1}{2}E\{x(\omega, t), y(\omega, t)\} = c_{xy}(\omega)\cos\theta + q_{xy}(\omega)\sin\theta.$$

The condition for a maximum, which is found by differentiating the function above with respect to θ and setting the result to zero, is $-c_{xy}(\omega)\sin\theta +$

$q_{xy}(\omega)\cos\theta = 0$. For the condition to be satisfied, there must be $\sin(\theta) = \alpha q_{xy}(\omega)$ and $\cos(\theta) = -\alpha c_{xy}(\omega)$ for some factor α. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, it follows that $\alpha = \{c^2_{xy}(\omega) + q^2_{xy}(\omega)\}^{-1/2}$, and there are

$$\text{(27)} \qquad \sin\theta = \frac{q_{xy}(\omega)}{\{c^2_{xy}(\omega) + q^2_{xy}(\omega)\}^{1/2}} \quad \text{and} \quad \cos\theta = \frac{-c_{xy}(\omega)}{\{c^2_{xy}(\omega) + q^2_{xy}(\omega)\}^{1/2}},$$

On substituting these into (26), it is found that

$$\text{(28)} \qquad c_{xy}(\omega)\cos\theta + q_{xy}(\omega)\sin\theta = \{c^2_{xy}(\omega) + q^2_{xy}(\omega)\}^{1/2}.$$

This maximised covariance measure is the numerator of the coherence function $\rho(\omega)$ of (24).

The function

$$\text{(29)} \qquad \theta(\omega) = \tan^{-1}\left\{\frac{q_{xy}(\omega)}{c_{xy}(\omega)}\right\}.$$

constitutes the phase spectrum of $x(t)$ and $y(t)$. It indicates, for each frequency, the extent to which the components of $y(t)$ lead or lag behind those of $x(t)$.

It has been shown that the measure of spectral coherence is unaffected by a changes in the phase of the sinusoidal components of which the processes are composed. It is also unaffected by systematic changes in their amplitudes. Thus, if the two processes $x(t)$ and $y(t)$ are transformed by linear filters and if the transfer functions of the filters have none of their poles or zeros on the unit circle, then their coherence is not affected.

Let $\alpha(L)$ and $\beta(L)$ be two filters and let

$$\text{(30)} \qquad \xi(t) = \alpha(L)x(t) \quad \text{and} \quad \zeta(t) = \beta(L)y(t).$$

Also let $\alpha(\omega) = \sum_j \alpha_j \exp(-\mathrm{i}\omega)$ and $\beta(\omega) = \sum_j \beta_j \exp(-\mathrm{i}\omega)$ be the corresponding frequency response functions. Then, the spectra of $\xi(t)$ and $\zeta(t)$ together with their co-spectrum are given by

$$\text{(31)} \qquad \begin{aligned} f^{\xi\xi}(\omega) = |\alpha(\omega)|^2 f^{xx}(\omega), \quad f^{\zeta\zeta}(\omega) = |\beta(\omega)|^2 f^{yy}(\omega) \\ \text{and} \quad g^{\xi\zeta}(\omega) = \alpha^*(\omega)\beta(\omega)g^{xy}(\omega). \end{aligned}$$

The modulus of the cross spectrum of $\xi(t)$ and $\zeta(t)$ is

$$\text{(32)} \qquad |g^{\xi\zeta}(\omega)| = |\alpha(\omega)||\beta(\omega)||g^{xy}(\omega)|.$$

Therefore, the coherence of the transformed processes is

$$\text{(33)} \qquad \rho^{\xi\zeta}(\omega) = \frac{|g^{\xi\zeta}(\omega)|}{\{f^{\xi\xi}(\omega)f^{\zeta\zeta}(\omega)\}^{1/2}} = \frac{|g^{xy}(\omega)|}{\{f^{xx}(\omega)f^{yy}(\omega)\}^{1/2}} = \rho^{xy}(\omega),$$

which is equal to the original coherence of $x(t)$ and $y(t)$.