LINEAR DIFFERENTIAL EQUATIONS

Consider the second-order linear homogeneous differential equation

$$(1) \qquad \rho_0 \frac{d^2 y(t)}{dt^2} + \rho_1 \frac{dy(t)}{dt} + \rho_2 y(t) = 0.$$

The general solution is given by

$$(2) \qquad y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

where λ_1, λ_2 are the roots of the auxiliary equation

$$(3) \qquad \begin{aligned} \rho_0 x^2 + \rho_1 x + \rho_2 &= \rho_0 (x - \lambda_1)(x - \lambda_2) \\ &= \rho_0 \big\{ x^2 - (\lambda_1 + \lambda_2)x + \lambda_1 \lambda_2 \big\} = 0. \end{aligned}$$

The roots are given by

$$(4) \qquad \lambda_1, \lambda_2 = \frac{-\rho_1 \pm \sqrt{\rho_1^2 - 4\rho_0\rho_2}}{2\rho_0}.$$

In the case where $\rho_1^2 < 4\rho_0\rho_2$, this becomes

$$(5) \qquad \begin{aligned} \lambda, \lambda^* &= \frac{-\rho_1 \pm i\sqrt{4\rho_0\rho_2 - \rho_1^2}}{2\rho_0} \\ &= \eta \pm i\omega; \end{aligned}$$

and the auxiliary equation can then be written as

$$(6) \qquad \rho_0 x^2 + \rho_1 x + \rho_2 = \rho_0 \big\{ x^2 - 2\eta x + (\eta^2 + \omega^2) \big\} = 0.$$

In the case of complex roots, the general solution assumes the form of

$$(7) \qquad \begin{aligned} y(t) &= ce^{(\eta + i\omega)t} + c^* e^{(\eta - i\omega)t} \\ &= e^{\eta t}\{ce^{i\omega t} + c^* e^{-i\omega t}\}. \end{aligned}$$

This is a real-valued sequence; and, since a real term must equal its own conjugate, we require c and c^* to be conjugate numbers of the form

$$(8) \qquad \begin{aligned} c^* &= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ c &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}. \end{aligned}$$

Thus we have

$$(9) \qquad \begin{aligned} y(t) &= \rho e^{\eta t}\{e^{i(\omega t - \theta)} + e^{-i(\omega t - \theta)}\} \\ &= 2\rho e^{\eta t}\cos(\omega t - \theta). \end{aligned}$$

Example. An idealised physical model of an oscillatory system consists of a weight of mass m suspended from a helical spring of negligible mass which exerts a force proportional to its extension. Let y be the displacement of the weight from its position of rest and let h be Hooke's modulus which is the force exerted by the spring per unit of extension. Then Newton's second law of motion gives the equation

$$m\frac{d^2y}{dt^2} + hy = 0. \tag{10}$$

This is an instance of a second-order differential equation. The solution is

$$y(t) = 2\rho\cos(\omega_n t - \theta), \tag{11}$$

where $\omega_n = \sqrt{h/m}$ is the so-called natural frequency and ρ and θ are constants determined by the initial conditions. There is no damping or frictional force in the system and its motion is perpetual.

In a system which is subject to viscous damping, the resistance to the motion is proportional to its velocity. The differenential equation becomes

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + hy = 0, \tag{12}$$

where c is the damping coefficient. The auxiliary equation of the system is

$$\begin{aligned} mx^2 + cx + h &= m(x - \lambda_1)(x - \lambda_2) \\ &= 0, \end{aligned} \tag{13}$$

and the roots are given by

$$\lambda_1, \lambda_2 = \frac{-c \pm \sqrt{c^2 - 4mh}}{2m}. \tag{14}$$

The character of the system's motion depends upon the discriminant $c^2 - 4mh$. If $c^2 < 4mh$, then the motion will be oscillatory, whereas, if $c^2 > 4mh$, the displaced weight will return to its position of rest without overshooting. If $c^2 = 4mh$, then the system is said to be critically damped. The critical damping coefficient is defined by

$$c_c = 2\sqrt{mh} = 2m\omega_n, \tag{15}$$

where ω_n is the natural frequency of the undamped system. On defining the so-called damping ratio $\zeta = c/c_c$ we may write equation (14) as

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{16}$$

In the case of light damping, where $\zeta < 1$, the equation of the roots becomes

$$(17)\qquad \begin{aligned}\lambda, \lambda^* &= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}\\ &= \eta \pm i\omega;\end{aligned}$$

and the motion of the system is given by

$$(18)\qquad \begin{aligned}y(t) &= 2\rho e^{\eta t}\cos(\omega t - \theta)\\ &= 2\rho e^{-\zeta\omega_n t}\cos\Big\{(1-\zeta^2)^{1/2}\omega_n t - \theta\Big\}.\end{aligned}$$

The damping ratio of an oscillatory system can be expressed in terms of the complex roots $\eta \pm i\omega$ of the auxiliary equation $mx^2 + cx + h = 0$ as well as in terms of the parameters of the equation. From

$$(19)\qquad \eta^2 = \zeta^2\omega_n^2 \quad \text{and} \quad \omega^2 = (1-\zeta^2)\omega_n^2,$$

we can deduce that

$$(20)\qquad \zeta^2 = \frac{\eta^2}{\eta^2 + \omega^2}.$$

LINEAR DIFFERENCE EQUATIONS

Consider the second-order linear homogeneous differential equation

$$(21)\qquad \alpha_0 y(t) + \alpha_1 y(t-1) + \alpha_2 y(t-2) = 0.$$

The general solution is given by

$$(22)\qquad y(t) = c_1\mu_1^t + c_2\mu_2^t,$$

where μ_1, μ_2 are the roots of the auxiliary equation

$$(23)\qquad \begin{aligned}\alpha_0 z^2 + \alpha_1 z + \alpha_2 &= \alpha_0(z-\mu_1)(z-\mu_2)\\ &= \alpha_0\big\{z^2 - (\mu_1+\mu_2)z + \mu_1\mu_2\big\} = 0.\end{aligned}$$

The roots are given by

$$(24)\qquad \mu_1, \mu_2 = \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0\alpha_2}}{2\alpha_0}.$$

In the case where $\alpha_1^2 < 4\alpha_0\alpha_2$, this becomes

$$(25)\qquad \begin{aligned} \mu, \mu^* &= \frac{-\alpha_1 \pm i\sqrt{4\alpha_0\alpha_2 - \alpha_1^2}}{2\alpha_0} \\ &= \gamma \pm i\delta; \end{aligned}$$

and the auxiliary equation can then be written as

$$(26)\qquad \alpha_0 z^2 + \alpha_1 z + \alpha_2 = \alpha_0\big\{z^2 - 2\gamma z + (\gamma^2 + \delta^2)\big\} = 0.$$

The complex roots can be written in three alternative ways:

$$(27)\qquad \begin{aligned} \mu &= \gamma + i\delta = \kappa(\cos\omega + i\sin\omega) = \kappa e^{i\omega}, \\ \mu^* &= \gamma - i\delta = \kappa(\cos\omega - i\sin\omega) = \kappa e^{-i\omega}. \end{aligned}$$

Here we have

$$(28)\qquad \kappa = \gamma^2 + \delta^2 \quad \text{and} \quad \omega = \tan^{-1}\left(\frac{\delta}{\gamma}\right).$$

The general solution of the difference equation in the case of complex root may be expressed as

$$(29)\qquad y(t) = c\mu^t + c^*(\mu^*)^t.$$

This is a real-valued sequence; and, since a real variable must equal its own conjugate, we require c and c^* to be conjugate numbers of the form

$$(30)\qquad \begin{aligned} c^* &= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}, \\ c &= \rho(\cos\theta - i\sin\theta) = \rho e^{-i\theta}. \end{aligned}$$

Thus we have

$$(31)\qquad \begin{aligned} c\mu^t + c^*(\mu^*)^t &= \rho e^{-i\theta}(\kappa e^{i\omega})^t + \rho e^{i\theta}(\kappa e^{-i\omega})^t \\ &= \rho\kappa^t\left\{e^{i(\omega t-\theta)} + e^{-i(\omega t-\theta)}\right\} \\ &= 2\rho\kappa^t\cos(\omega t - \theta). \end{aligned}$$

It may be useful to express some of the parameters of this equation in terms of the coefficients of the original difference equation under (21). Thus we can find that

$$(32)\qquad \omega = \tan^{-1}\left(\frac{\sqrt{4\alpha_0\alpha_2 - \alpha_1^2}}{\alpha_1}\right) \quad \text{and} \quad \kappa = \frac{\alpha_2}{\alpha_0}.$$