CONVERSION OF FILTERS FROM LOWPASS TO BANDPASS

The idea of a frequency transformation is to derive a filter from an existing prototype in a way that preserves some of its frequency-domain characteristics. The prototype is usually a lowpass or a highpass filter and the derived filters are typically bandpass filters or bandstop filters.

A highpass filter can be derived by subtracting a lowpass filter from an allpass filter or from unity. Likewise, a lowpass filter can be derived from a highpass filter by subtraction. A pair of filters that sum to unity are described as complementary. As well as complementary highpass and lowpass filters, there are also complementary pairs of bandpass and bandstop filters.

We shall concentrate our attention on a lowpass prototype filter and we shall consider transforming it to another lowpass filter and to a bandpass filter. Highpass filters and bandstop filters which are complementary to these filters may be found by subtracting them from unity. There is also a simple reflection in the frequency domain around the point $\omega = \pi/2$, which divides the positive frequency range, that serves to convert a lowpass filter to a highpass filter.

A filter may be characterised by it impulse response $\{\psi_j; j = 0 \pm 1, \pm 2, \ldots\}$, which may be a finite or an infinite sequence. In the case of an finite impulse response (FIR) filter, the sequence is synonymous with the filter coefficients. In the case of an infinite impulse response (IIR) filter, the impulse response is often generated by taking the coefficients from the series expansion of a rational operator. Regardless of the origin of the sequence, we shall denote its z-transform by

$$\psi(z) = \sum_j \psi_j z^j, \tag{1}$$

and this notation can also serve equally for the rational function $\psi(z) = \mu(z)/\alpha(z)$ or for its series expansion, which will correspond to the RHS of the equation above.

Setting $z = e^{-\mathrm{i}\omega}$ gives the frequency response of the filter, which may be denoted either by $\psi(e^{\mathrm{i}\omega})$ or, more economically, by $\psi(\omega)$. In general, the frequency response $\psi(\omega)$ is a complex-valued periodic function of ω with a period of 2π. The squared modulus of the filter is

$$|\psi(z)|^2 = \psi(z^{-1})\psi(z), \tag{2}$$

and setting $z = e^{-\mathrm{i}\omega}$ gives the squared gain of the filter.

Since the filter coefficients are real-valued, the frequency response function, which is their Fourier transform, is symmetric about the point $\omega = 0$ of zero frequency. For any value of ω, it maps to a point in the complex plane. The function can be represented in polar form by

$$\psi(\omega) = |\psi(\omega)|e^{-\mathrm{i}\theta(\omega))} = |\psi(\omega)|[\cos\{\theta(\omega)\} - \mathrm{i}\sin\{\theta(\omega)\}], \tag{3}$$

where $|\psi(\omega)|$ is the distance of the point from the origin and $\mathrm{Arg}\{\psi(\omega)\} = -\theta(\omega)$ is the angle that the line joining the point to the origin makes with the horizontal real axis.

Frequency Shifting

The first frequency transformation to consider is one that creates as bandpass filter from a lowpass filter. The frequency response of the ideal lowpass filter, defined on the interval $[-\pi, \pi]$, is given by

$$\psi(\omega) = \begin{cases} 1, & \text{if } \omega \in (-\delta, \delta); \\ 1/2, & \text{if } \omega = \pm\delta, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Since the cut-off points $\pm\delta$ constitute a set of measure zero in the context of the frequency interval, it is common to neglect to specify the corresponding ordinates $\psi(-\delta) = \psi(\delta) = 1/2$. The coefficients of this filter are given by the inverse Fourier transform of the frequency response:

$$\psi_j = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-\mathrm{i}\omega j} = \begin{cases} \dfrac{\delta}{\pi}, & \text{if } j = 0, \\ \dfrac{\sin(\delta j)}{\pi j}, & \text{if } j \neq 0. \end{cases} \tag{5}$$

To convert the lowpass filter to a bandpass filter, two copies are made of the lowpass passband and the copies are shifted so that their new centres lie at the frequencies $-\gamma$ and γ. The frequency response of the copy centred on γ is

$$\begin{aligned} \psi(\omega - \gamma) &= \sum_j \psi_j e^{-\mathrm{i}(\omega-\gamma)j} \\ &= \sum_j \psi_j \{e^{-\mathrm{i}\gamma j}\} e^{-\mathrm{i}\omega j}. \end{aligned} \tag{6}$$

On its own, this would generate a set of complex-valued filter coefficients. The other the copy, $\psi(\omega+\gamma)$, which must accompany it, is centred on $-\gamma$. It follows that the frequency response of the bandpass filter is

$$\begin{aligned} \psi_B(\omega) &= \psi(\omega - \gamma) + \psi(\omega + \gamma) \\ &= 2\sum_j \left\{ \psi_j \left(\frac{e^{\mathrm{i}\gamma j} + e^{-\mathrm{i}\gamma j}}{2} \right) \right\} e^{-\mathrm{i}\omega j} \\ &= 2\sum_j \{\psi_j \cos(\gamma j)\} e^{-\mathrm{i}\omega j}. \end{aligned} \tag{7}$$

The coefficients of the bandpass filter are the real-valued elements $\psi_j \cos(\omega j)$; and it may be said that they are obtained by modulating the original coefficients by a cosine function.

When the ideal filter is transformed in this way, the resulting filter has the response

$$\psi_B(\omega) = \begin{cases} 1, & \text{if } |\omega| \in (\alpha, \beta), \\ 1/2, & \text{if } \omega = \pm\alpha, \pm\beta, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where $\alpha = \gamma - \delta$ and $\beta = \gamma + \delta$ are the cut-off points of the pass band in the interval $[0, \pi]$. The bandpass coefficients that are derived from the ideal specification of (5) may be expressed variously as

$$\begin{aligned} 2\psi_j \cos(\gamma j) &= \frac{2}{\pi j} \sin(\delta j) \cos(\gamma j) \\ &= \frac{1}{\pi j} \sin\{(\gamma + \delta) j\} - \sin\{(\gamma - \delta) j\} \\ &= \frac{1}{\pi j} \{\sin(\beta) j - \sin(\alpha) j\}, \end{aligned} \tag{9}$$

where the trigonometrical identity $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$ accounts for second equality.

The first equality of (9) relates to the modulation of the coefficients of the prototype filter. The final equality, which may be written as $\psi_B(\omega) = \psi_\beta(\omega) - \psi_\alpha(\omega)$, indicates that the bandpass filter may be formed by the subtracting a lowpass filter with a cut-off at $\omega = \alpha$ from the a lowpass filter with a cut-off at $\omega = \beta$. Therefore, there are two alternative but equivalent ways of forming this bandpass filter.

In this example, the component lowpass filters $\psi_\beta(\omega)$ and $\psi_\alpha(\omega)$ originate as transformations of the same prototype filter $\psi(\omega)$. This is an unnecessary restriction, and, in general, the component filters may have distinct origins.

Allpass Frequency Transformations

The method of frequency shifting is not readily applicable to an IIR filter in the form of a rational transfer function, unless the latter is expanded as a power series. An alternative way of creating a bandpass filter from lowpass prototype, which is more flexible but more complicated, depends upon applying a so-called allpass transformation $g(z)$ to the argument z of the prototype function $\psi(z)$ to create a compound function $\psi_T(z) = \psi\{g(z)\}$.

The function $g(z)$ is designed to map the unit circle into itself. As z travels around the circle, $g(z)$ will travel the same path at a different rate and, possibly, from a different starting point. Moreover, depending on its specification, when z completes one circuit, $g(z)$ will complete one or more circuits.

If $g(z)$ completes a single circuit, then the local retardation or acceleration of its trajectory can be used to distort the frequency response of the prototype filter and to shift its location. If $g(z)$ completes a number of circuits, then the effect will be to map multiple copies of the filter response into a frequency interval of 2π.

The general specification of the all pass function is

$$g(z) = \pm \frac{\tilde{c}(z^{-1})}{c(z)} = \pm \frac{z^n c(z^{-1})}{c(z)} = \frac{z^n + c_1 z^{n-1} + \cdots + c_{n-1} z + c_n}{1 + c_1 z + \cdots + c_{n-1} z^{n-1} + c_n z^n}. \tag{10}$$

For the compound filter to be stable, it is necessary that all of the roots of $c(z)$ should lie outside the unit circle, Since the roots of $z^n c(z^{-1})$ are the reciprocals of the roots of $c(z)$, they must all lie inside the unit circle.

According to the argument principle, the number of times that the trajectory of a function $g(z)$ encircles the origin as $z = e^{i\omega}$ travels around the unit circle is equal to the number of zeros within the circle less the number of poles that lie within it. Therefore, if $c(z)$ is of degree n, then $g(z)$ will encircle the origin n times. We shall consider only cases where $n = 1, 2$.

It is straightforward to confirm the following properties:

$$\begin{aligned} &\text{(i)} \qquad g(z^{-1})g(z) = 1, \\ &\text{(ii)} \qquad \frac{\tilde{c}(1)}{c(1)} = 1 \quad \text{and} \quad \frac{\tilde{c}(-1)}{c(-1)} = -1. \end{aligned} \tag{11}$$

The first condition confines the trajectory of $g(z)$ to the unit circle. The second condition indicates that, if $\text{Arg}(z) = 0$, then $\text{Arg}\{\tilde{c}(z)/c(z)\} = 0$ whereas, if $\text{Arg}(z) = \pi$, then $\text{Arg}\{\tilde{c}(z)/c(z)\} = \pi$. That to say, the trajectories of z and $g(z) = \tilde{c}(z)/c(z)$ meet where the unit circle intersects the horizontal real axis. However, if a negative sign is applied to $\tilde{c}(z)/c(z)$, then $g(z)$ will be diametrically opposite to z when the two are on the real axis.

The coefficients of the allpass function may be determined so as to ensure an appropriate correspondence between the features of the prototype filter and those of the derived filter. The usual requirement is for an particular placement of the cut-off points of the pass bands of the derived filter. First, we shall consider a first-order transformation that shifts the cut-off point of a lowpass filter. Then, we shall consider a second-order transformation that converts a lowpass filter to a bandpass filter.

The first-order transformation is

$$g(z) = \pm \frac{z(1 - cz^{-1})}{1 - cz}. \tag{12}$$

When the sign is positive, $g(z)$ and z coincide at $z = \pm 1$. Elsewhere, $g(z)$ leads z or lags behind it consistently. If $c > 0$, then the effect of the lead of g will be to shift the cut-off point to the left to a lower frequency. If $c < 0$ then the effect of the lag of g will be to shift the cut off point to the right. If $c = 0$, then $g(z) = z$ and there will be no effect.

Consider also the transformation $g(z) = -z$, and let $z = e^{i\omega}$. Then, the effect of replacing z in $\psi(z)$ by $g(z)$ is to shift the frequency response by delaying or advancing it by π radians, which is half the period of this function. Thus, the lowpass response of the prototype filter over the interval $[-\pi, 0]$, which is

$$\psi(\omega) = \begin{cases} 0, & \text{for } \omega \in [-\pi, -\delta), \\ 1, & \text{for } \omega \in (-\delta, 0], \end{cases}$$

becomes

$$\psi_H(\omega) = \psi(\omega + \pi) = \begin{cases} 0, & \text{for } \omega \in [0, \pi - \delta); \\ 1, & \text{for } \omega \in (\pi - \delta, \pi], \end{cases}$$

which is the response of a highpass filter.

Now consider the case where it is desired to shift the lowpass cut-off frequency in $[0,\pi]$ from $\omega=\delta$ to $\omega=\kappa$. Then, $z=e^{\mathrm{i}\kappa}$ must be mapped into $g(z)=e^{\mathrm{i}\delta}$. To find the appropriate value of the allpass parameter h, we may write the function of (12) in the form of $g(1-cz)-z(1-cz^{-1})=0$. Then, multiplying by $g^{-1/2}z^{-1/2}$ gives

$$\begin{aligned}0&=g^{1/2}z^{-1/2}(1-cz)-g^{-1/2}z^{1/2}(1-cz^{-1})\\&=(g^{1/2}z^{-1/2}-g^{-1/2}z^{1/2})-c(g^{1/2}z^{1/2}-g^{-1/2}z^{-1/2}).\end{aligned}\tag{13}$$

Setting $z=e^{\mathrm{i}\kappa}$ and $g=e^{\mathrm{i}\delta}$ and dividing by 2i gives

$$0=\sin\left(\frac{\delta-\kappa}{2}\right)-c\sin\left(\frac{\delta+\kappa}{2}\right),\tag{14}$$

which serves to determine the coefficient c of the allpass transformation.

The second-order allpass function is

$$g(z)=\pm\frac{z^2(1+c_1z^{-1}+c_2z^{-2})}{1+c_1z+c_2z^2}.\tag{15}$$

which can be rewritten in a symmetrised homogeneous form as

$$(z^{-1}g^{1/2}\pm zg^{-1/2})+c_1(g^{1/2}\pm g^{-1/2})+c_2(g^{1/2}z\pm g^{-1/2}z^{-1})=0.\tag{16}$$

With $z=e^{\mathrm{i}\kappa}$ and $g=e^{\mathrm{i}\delta}$, the expression can be rendered in terms or cosine functions, if the expressions in parentheses are sums, or in terms of sines if the expressions are differences.

Consider the case of the conversion of a lowpass filter to a bandpass filter. The trajectory of g must travel twice around the unit circle for each cycle of z. In the process, two images of the response of the lowpass prototype filter over $[-\pi,\pi]$ will be mapped into the interval $[0,2\pi]$. The first image will be mapped into $[0,\pi]$, which is the positive frequency interval of the derived bandpass filter. The second image will be mapped into $[\pi,2\pi]$, which is synonymous with the negative frequency interval $[-\pi,\pi]$ of the derived filter.

Since g starts its trajectory at 0 when z starts at $-\pi$, there is a phase lag of π radians, which implies that the sign on $g(z)$ must be is negative. Therefore, in this case, the expressions in the parentheses of (16) are sine functions. To achieve the desired locations of the bandpass cut-off points, which are at the frequencies $\alpha,\beta\in[0,\pi]$, the point $z=e^{-\mathrm{i}\delta}$ must be mapped into $z=e^{-\mathrm{i}\alpha}$ and the point $z=e^{\mathrm{i}\delta}$ must be mapped into $z=e^{\mathrm{i}\beta}$. By substituting these pairs of points into (16), we may derive two linear equations that can be solved for the values of the parameters c_1,c_2 of the allpass function $g(z)$.

Given the prolific nature of the trigonometrical identities, there are numerous ways in which the solutions to these equations may be presented. The conventional expression for the solution is due to Constantindes (1967–1970):

$$g(z)=-\frac{z^2-\dfrac{2\alpha k}{k+1}z+\dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1}z^2+\dfrac{2\alpha k}{k+1}z+1},\qquad \begin{aligned}\alpha&=\frac{\cos\{(\beta+\alpha)/2\}}{\cos\{\beta-\alpha)/2\}},\\ k&=\cot\left(\frac{(\beta-\alpha}{2}\right)\tan\left(\frac{\delta}{2}\right).\end{aligned}\tag{17}$$

References

Constantinides, A.G., (1967), Frequency Transformations for Digital Filters, *Electronic Letters,* 3, 487–489.

Constantinides, A.G., (1968), Frequency Transformations for Digital Filters, *Electronic Letters,* 4, 115–116.

Constantinides, A.G., (1969), Design of Bandpass Digital Filters, *Proc. IEEE,* 1, 1129–1231. 115–116.

Constantinides, A.G., (1970), Spectral Transformations for Digital Filters, *IEE Proc, Inst. Elect. Eng,* 117, 1585–1590.