## THE INTEGRATED MOVING-AVERAGE MODEL IMA(1, 1)

Consider the case of a first-order random walk of which the observations are subject to an additive white-noise error. The equation of the random walk is

$$\xi(t) = \xi(t-1) + \nu(t), \tag{1}$$

and the equation of the observations is

$$y(t) = \xi(t) + \eta(t). \tag{2}$$

It is assumed that the white-noise processes $\nu(t)$ and $\eta(t)$ are mutually independent. The random walk may also be expressed as

$$(1-L)\xi(t) = \nu(t). \tag{3}$$

Combining this with the observation equation (2) gives

$$(1-L)y(t) = \nu(t) + (1-L)\eta(t). \tag{4}$$

It is straightforward to show that the terms of the final expression combine to give rise to a first-order moving-average process. Let this expression be denoted by

$$q(t) = \nu(t) + \eta(t) - \eta(t-1). \tag{5}$$

Then the variance and the first autocovariance of this sequence are given by

$$\begin{aligned} V(q_t) &= \sigma_\nu^2 + 2\sigma_\eta^2 = \gamma_0, \\ C(q_t, q_{t-1}) &= -\sigma_\eta^2 = \gamma_1. \end{aligned} \tag{6}$$

The autocovariances at higher lags are all zero-valued. Now consider the equation which relates the autocovariances $\gamma_0, \gamma_1$ of the MA(1) process $q(t) = (I - \theta L)\varepsilon(t)$ to its parameters $\theta$ and $\sigma_\varepsilon^2 = V\{\varepsilon_t\}$:

$$\begin{aligned} \gamma_0 &= \sigma_\varepsilon^2(1+\theta^2), \\ \gamma_1 &= -\sigma_\varepsilon^2\theta. \end{aligned} \tag{7}$$

The ratio of these equations gives the following expression for the autocorrelation coefficient:

$$\rho = \frac{\gamma_1}{\gamma_0} = \frac{-\theta}{1+\theta^2}. \tag{8}$$

This leads to a quadratic equation in the form of $\theta^2\rho + \theta + \rho = 0$ of which the solution is

$$\theta = \frac{-1 \pm \sqrt{1-4\rho^2}}{2\rho}. \tag{9}$$

The solution is real-valued if and only if $|\rho| \leq \frac{1}{2}$. The latter condition is manifestly satisfied by the value of $\rho = \gamma_1/\gamma_0$ which is formed from the elements under (6). Thus it follows that the composite process $q(t)$ depicted in equation (5) can be expressed as

$$q(t) = (I - \theta L)\varepsilon(t). \tag{10}$$

Therefore the combination of equations (1) and (2) is an integrated moving-average IMA(1, 1) in the form of

$$(I - L)y(t) = (I - \theta L)\varepsilon(t). \tag{11}$$

## FORECASTING THE IMA(1, 1) PROCESS VIA EXPONENTIAL SMOOTHING

Equation (11) can be written as

$$y(t) = y(t-1) + \varepsilon(t) - \theta\varepsilon(t-1). \tag{12}$$

It is easy to see that the forecasts of the process made at time $t$ are as follows:

$$\begin{aligned} \hat{y}_{t+1|t} &= y_t - \theta\varepsilon_t, \\ \hat{y}_{t+2|t} &= \hat{y}_{t+1|t}, \\ &\vdots \\ \hat{y}_{t+h|t} &= \hat{y}_{t+h-1|t}. \end{aligned} \tag{13}$$

Thus, the forecasting rule is to extrapolate the value of the one-step-ahead forecast into the indefinite future.

It remains to find an expression for the one-step-ahead forecast which is in terms of previous values of the observable sequence $y(t)$. Therefore consider rearranging equation (11) to give

$$\begin{aligned} \varepsilon(t) &= \frac{1-L}{1-\theta L}y(t) \\ &= (1-L)\Big\{1 + \theta L + \theta^2 L^2 + \cdots\Big\}y(t) \\ &= \Big[1 - (1-\theta)\big\{L + \theta L^2 + \theta^2 L^3 + \cdots\big\}\Big]y(t). \end{aligned} \tag{14}$$

This gives

$$y(t) = (1-\theta)\Big\{y(t-1) + \theta y(t-2) + \theta^2 y(t-3) + \cdots\Big\} + \varepsilon(t). \tag{15}$$

It follows that the expression for the one-step-ahead forecast is

$$\hat{y}_{t+1|t} = (1-\theta)\big\{y_t + \theta y_{t-1} + \theta^2 y_{t-2} + \cdots\big\}; \tag{16}$$

and this is just the formula for exponential smoothing.