THE EQUATIONS OF THE KALMAN FILTER

The state-space model, which underlies the Kalman filter, consists of two equations

$$y_t = H_t\xi_t + \eta_t, \qquad \textit{Observation Equation} \tag{1}$$

$$\xi_t = \Phi_t\xi_{t-1} + \nu_t, \qquad \textit{Transition Equation} \tag{2}$$

where y_t is the observation on the system and ξ_t is the state vector. The observation error η_t and the state disturbance ν_t are mutually uncorrelated random vectors of zero mean with dispersion matrices

$$D(\eta_t) = \Omega_t \qquad \text{and} \qquad D(\nu_t) = \Psi_t. \tag{3}$$

It is assumed that the matrices H_t, Φ_t, Ω_t and Ψ_t are known for all $t = 1, \ldots, n$ and that an initial estimate x_0 is available for the state vector ξ_0 at time $t = 0$ together with a dispersion matrix $D(\xi_0) = P_0$. The empirical information available at time t is the set of observations $\mathcal{I}_t = \{y_1, \ldots, y_t\}$.

The Kalman-filter equations determine the state-vector estimates $x_{t|t-1} = E(\xi_t|\mathcal{I}_{t-1})$ and $x_t = E(\xi_t|\mathcal{I}_t)$ and their associated dispersion matrices $P_{t|t-1}$ and P_t. From $x_{t|t-1}$, the prediction $\hat{y}_{t|t-1} = H_t x_{t|t-1}$ is formed which has a dispersion matrix F_t. A summary of these equations is as follows:

$$x_{t|t-1} = \Phi_t x_{t-1}, \qquad \textit{State Prediction} \tag{4}$$

$$P_{t|t-1} = \Phi_t P_{t-1}\Phi_t' + \Psi_t, \qquad \textit{Prediction Dispersion} \tag{5}$$

$$e_t = y_t - H_t x_{t|t-1}, \qquad \textit{Prediction Error} \tag{6}$$

$$F_t = H_t P_{t|t-1} H_t' + \Omega_t, \qquad \textit{Error Dispersion} \tag{7}$$

$$K_t = P_{t|t-1} H_t' F_t^{-1}, \qquad \textit{Kalman Gain} \tag{8}$$

$$x_t = x_{t|t-1} + K_t e_t, \qquad \textit{State Estimate} \tag{9}$$

$$P_t = (I - K_t H_t)P_{t|t-1}. \qquad \textit{Estimate Dispersion} \tag{10}$$

Alternative expressions are available for P_t and K_t are available on the assumption that Ω_t is nonsingular:

$$P_t = (P_{t|t-1}^{-1} + H_t'\Omega_t^{-1}H_t)^{-1}, \tag{11}$$

$$K_t = P_t H_t'\Omega_t^{-1}. \tag{12}$$

By applying the well-known matrix inversion lemma to the expression on the RHS of (11), we obtain the original expression for P_t given under (10). To verify the identity $P_{t|t-1}H_t'F_t^{-1} = P_t H_t'\Omega_t^{-1}$ which equates (8) and (12), we write it as $P_t^{-1}P_{t|t-1}H_t' = H_t'\Omega_t^{-1}F_t$. The latter is readily confirmed using the expression for P_t from (11) and the expression for F_t from (7).

Derivation of the Kalman Filter. The equations of the Kalman filter may be derived using the ordinary algebra of conditional expectations which indicates that, if x, y are jointly distributed variables which bear the linear relationship $E(y|x) = \alpha + B\{x - E(x)\}$, then

$$E(y|x) = E(y) + C(y,x)D^{-1}(x)\big\{x - E(x)\big\}, \tag{13}$$

$$D(y|x) = D(y) - C(y,x)D^{-1}(x)C(x,y), \tag{14}$$

$$E\big\{E(y|x)\big\} = E(y), \tag{15}$$

$$D\big\{E(y|x)\big\} = C(y,x)D^{-1}(x)C(x,y), \tag{16}$$

$$D(y) = D(y|x) + D\big\{E(y|x)\big\}, \tag{17}$$

$$C\big\{y - E(y|x), x\big\} = 0. \tag{18}$$

Of the equations listed under (4)—(10), those under (6) and (8) are merely definitions.

To demonstrate equation (4), we use (15) to show that

$$\begin{aligned} E(\xi_t|\mathcal{I}_{t-1}) &= E\big\{E(\xi_t|\xi_{t-1})|\mathcal{I}_{t-1}\big\} \\ &= E\big\{\Phi_t\xi_{t-1}|\mathcal{I}_{t-1}\big\} \\ &= \Phi_t x_{t-1}. \end{aligned} \tag{19}$$

We use (17) to demonstrate equation (5):

$$\begin{aligned} D(\xi_t|\mathcal{I}_{t-1}) &= D(\xi_t|\xi_{t-1}) + D\big\{E(\xi_t|\xi_{t-1})|\mathcal{I}_{t-1}\big\} \\ &= \Psi_t + D\big\{\Phi_t\xi_{t-1}|\mathcal{I}_{t-1}\big\} \\ &= \Psi_t + \Phi_t P_{t-1}\Phi_t'. \end{aligned} \tag{20}$$

To obtain equation (7), we substitute (1) into (6) to give $e_t = H_t(\xi_t - x_{t|t-1}) + \eta_t$. Then, in view of the statistical independence of the terms on the RHS, we have

$$\begin{aligned} D(e_t) &= D\big\{H_t(\xi_t - x_{t|t-1})\big\} + D(\eta_t) \\ &= H_t P_{t|t-1} H_t' + \Omega_t = D(y_t|\mathcal{I}_{t-1}). \end{aligned} \tag{21}$$

To demonstrate the updating equation (9), we begin by noting that

$$\begin{aligned} C(\xi_t, y_t|\mathcal{I}_{t-1}) &= E\big\{(\xi_t - x_{t|t-1})y_t'\big\} \\ &= E\big\{(\xi_t - x_{t|t-1})(H_t\xi_t + \eta_t)'\big\} \\ &= P_{t|t-1}H_t'. \end{aligned} \tag{22}$$

It follows from (13) that

$$\begin{aligned} E(\xi_t|\mathcal{I}_t) &= E(\xi_t|\mathcal{I}_{t-1}) + C(\xi_t, y_t|\mathcal{I}_{t-1})D^{-1}(y_t|\mathcal{I}_{t-1})\big\{y_t - E(y_t|\mathcal{I}_{t-1})\big\} \\ &= x_{t|t-1} + P_{t|t-1}H_t'F_t^{-1}e_t. \end{aligned} \tag{23}$$

The dispersion matrix under (10) for the updated estimate is obtained via equation (14):

$$\begin{aligned} D(\xi_t|\mathcal{I}_t) &= D(\xi_t|\mathcal{I}_{t-1}) - C(\xi_t, y_t|\mathcal{I}_{t-1})D^{-1}(y_t|\mathcal{I}_{t-1})C(y_t, \xi_t|\mathcal{I}_{t-1}) \\ &= P_{t|t-1} - P_{t|t-1}H_t'F_t^{-1}H_tP_{t|t-1}. \end{aligned} \tag{24}$$