Martingales

A martingale, according to the original meaning, is a particular gambling strategy in which the player doubles his wager at every loss until he wins or exhausts his funds.

Imagine that the gambler wagers £1 with a 50–50 chance of winning or loosing. If he looses, he wagers £2 on the next play. If he loses the nth play, then he wagers £2^n on the next play. If the gambler is wealthy enough to follow this strategy in spite of the losses that he might sustain, he will eventually cover all of his losses in his final and (virtually) inevitable win. The strategy is usually prohibited by casinos; and the croupiers are instructed to refuse the bets of anyone who seems to be pursuing it.

The term martingale has acquired a more general meaning in the context of mathematical statistics; but this meaning retains some of the flavour of the original example. To formalise the example, let us imagine that the gambler begins with an initial capita of S_0. By the tth play, his capital will have diminished to the value of S_t. Given that he faces a fair gain with equal prospects of winning and losing, he can form no expectation of his future position other than by projecting his present position forwards in time. Thus

$$E(S_{t+1}|S_t,\ldots,S_0,S_1,)=S_t.$$

The following definition represents an evident generalisation of this circumstance:

> A sequence $\{S_t,S_{t-1},\ldots,S_0\}$ is a martingale with respect to the (information) sequence $\{X_t,X_{t-1},\ldots,X_0\}$ if, for all $t\geq 1$, there is
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> $$\text{(a)}\quad E(S_t)<\infty \qquad \text{and} \qquad \text{(b)}\quad E(S_{t+1}|X_t,X_{t-1},\ldots,X_0)=S_t.$$

Example. Let $\{X_1,X_2,\ldots\}$ be a sequence of independently distributed random variables with $E(X_t)=0$ for all t. Then the sequence of the partial sums $S_t=X_1+X_2+\cdots+X_t$ is a martingale with respect to $\{X_t\}$ and there is

$$\begin{aligned}E(S_{t+1}|X_t,X_{t-1},\ldots,X_1\} &= E(S_t+X_{t+1}|X_t,X_{t-1},\ldots,X_1\}\\ &= E(S_t|X_t,X_{t-1},\ldots,X_1\}+E(X_{t+1}|X_t,X_{t-1},\ldots,X_1\}\\ &= S_t+0,\end{aligned}$$

which follows from the assumption of independence.

If we were to assume that $\{X_t\}$ is a sequence of independently and identically distributed random variables, then $\{S_t\}$ would become an ordinary random walk. It has been proposed, in the past, that certain share prices follow random walks. In that case, they would conform to the efficient markets hypothesis which asserts that the movements of the series are not predicatable from the past history of the share index. However, the assumption that the increments of the index are identically distributed is too strong. Thus, for example, the increments might be modelled by a GARCH process in which the variance is liable to change over time. The efficient markets hypothesis, which would also be fulfilled by a share index following a GARCH process, is sometimes described as the martingale hypothesis.