MA Processes with Common Autocovariances

It is easy to see that more than one moving-average process can share the same autocovariance function. Thus the equations under (1) on the sheets concerning *Computing the MA Parameters* are generated not only by the process $y(t) = \mu_0\varepsilon(t) + \mu_1\varepsilon(t-1) + \cdots + \mu_q\varepsilon(t-q)$ but also by the process $y(t) = \mu_q\varepsilon(t) + \mu_{q-1}\varepsilon(t-1) + \cdots + \mu_0\varepsilon(t-q)$, which is formed by reversing the sequence of coefficients, and by the process $y(t) = \mu_0\varepsilon(t) + \mu_1\varepsilon(t+1) + \cdots + \mu_q\varepsilon(t+q)$ in which the direction of time has been reversed. The latter process can also be formed from the second process by postdating $y(t)$ by q periods.

In order to characterise fully the set of processes which correspond to a given sequence of autocovariances, we must look more closely at the structure of the autocovariance generating function. Therefore let us factorise $\mu(z)$ in various ways so as to obtain

$$\mu(z) = \mu_0 \prod_{i=1}^{q}(1 - \frac{z}{\lambda_i}) \qquad \text{and}$$
$$\mu(z^{-1}) = \mu_n z^{-q} \prod_{i=1}^{q}\bigl(1 - z\lambda_i\bigr) \tag{1}$$

where

$$\mu_n = \mu_0 \prod_{i=1}^{q}(-\lambda_i)^{-1}. \tag{2}$$

These forms enable us to express the autocovariance generating function of the process $y(t) = \mu(L)\varepsilon(t)$ in a way which reveals its dependence on the roots of the polynomial equation $\mu(z) = 0$:

$$\gamma(z) = \sigma_\varepsilon^2 \mu_0^2 z^{-q} \Bigl\{\prod_{i=1}^{q}(-\lambda_i)^{-1}\Bigr\}\Bigl\{\prod_{i=1}^{q}(1 - \frac{z}{\lambda_i})(1 - z\lambda_i)\Bigr\}. \tag{3}$$

The braces in this expression help to separate the important factors. The first braces contains the product of the roots multiplied by $(-1)^n$. The second braces contains an expression which is unchanged when any root λ_i is replaced by its inverse $1/\lambda_i$. Let us now consider an new process $y_*(t) = \mu_*(L)\varepsilon_*(t)$ wherein

$$\mu_*(z) = \mu_0 \prod_{i=1}^{m}(1 - z\lambda_i) \prod_{i=m+1}^{q}(1 - \frac{z}{\lambda_i}) \tag{4}$$

and

$$V\bigl\{\varepsilon_*(t)\bigr\} = \sigma_\varepsilon^2 \prod_{i=1}^{m}|\lambda_i|^{-2}. \tag{5}$$

This process has been derived by replacing m of the roots of $\mu(z)$ by their inverses and by rescaling the variance of $\varepsilon(t)$. It is easy to see that its generating function $\gamma_*(z) = \sigma_{\epsilon_*}^2\mu_*(z)\mu_*(z^{-1})$ is the same as the function generating function $\gamma(z)$ of the original process. Clearly, we can can invert an arbitrary selection of the roots of $\mu(z)$ in this way and still retain the same autocovariance generating function. By taking account of all such inversions, we can define the complete class of processes that share the common autocovariance function. Amongst such a class there can be no more than one process which satisfies the condition of stationarity which requires every root of $\mu(z) = 0$ to lie outside the unit circle.

As an example, consider the MA(2) process which is specified by

$$\begin{aligned} y(t) &= (\mu_0 + \mu_1 L + \mu_2 L^2)\varepsilon(t) \\ &= (1 + 5L + 6L^2)\varepsilon(t), \end{aligned} \tag{6}$$

where $\sigma_\varepsilon^2 = 1$. The autocovariances of this process are given by

$$\begin{aligned} \gamma_0 &= \sigma_\varepsilon^2(\mu_0^2 + \mu_1^2 + \mu_2^2), \\ \gamma_1 &= \sigma_\varepsilon^2(\mu_0\mu_1 + \mu_1\mu_2), \\ \gamma_2 &= \sigma_\varepsilon^2\mu_0\mu_2; \end{aligned} \tag{7}$$

and , with $\mu_0 = 1$, $\mu_1 = 5$ and $\mu_2 = 6$ together with $\sigma_\varepsilon^2 = 1$, we get $\gamma_0 = 62$, $\gamma_1 = 30$ and $\gamma_2 = 6$.

Now consider the factorisation

$$\begin{aligned} \mu(z) &= (1 + 5z + 6z^2) \\ &= (1 + 3z)(1 + 2z). \end{aligned} \tag{8}$$

This shows that $\lambda_1 = -1/3$ and $\lambda_2 = -1/2$, and both of these are inside the unit circle, which violates the condition of invertibility. To obtain an invertible model which generates the same autcovariances, we simply invert the roots so as to give

$$\begin{aligned} \mu_*(z) &= \left(1 + \frac{1}{3}z\right)\left(1 + \frac{1}{2}z\right) \\ &= \left(1 + \frac{5}{6}z + \frac{1}{6}z^2\right) \end{aligned} \tag{9}$$

and

$$V\{\varepsilon_*(t)\} = \frac{\sigma_\varepsilon^2}{|\lambda_1\lambda_2|} = 6. \tag{10}$$