The Poles and Zeros of the Filter

The characteristics of a linear filter $\psi(L) = \delta(L)/\gamma(L)$, which are manifested in its frequency-response function, can be explained in terms of the location in the complex plane of the poles and zeros of $\psi(z^{-1}) = \delta(z^{-1})/\gamma(z^{-1})$ which include the roots of the constituent polynomials $\gamma(z^{-1})$ and $\delta(z^{-1})$. Consider therefore the expression

$$\psi(z^{-1}) = z^{g-d}\frac{\delta_0 z^d + \delta_1 z^{d-1} + \cdots + \delta_d}{\gamma_0 z^g + \gamma_1 z^{g-1} + \cdots + \gamma_g}. \tag{1}$$

This stands for a causal or backward-looking filter. In fact, the restriction of causality is unnecessary, and the action of the filter can be shifted in time without affecting its essential properties. Such an shift would be represented by multiplying the filter by a power of z. There would be no effect upon the gain of the filter, whilst the effect upon the phase would be linear, in the sense that each component of a signal, regardless of its frequency, would be advanced (if the power were positive) or delayed (if the power were negative) by the same amount of time.

The numerator and denominator of $\psi(z^{-1})$ may be factorised to give

$$\psi(z^{-1}) = z^{g-d}\frac{\delta_0}{\gamma_0}\frac{(z-\mu_1)(z-\mu_2)\cdots(z-\mu_d)}{(z-\kappa_1)(z-\kappa_2)\cdots(z-\kappa_g)}, \tag{2}$$

where $\mu_1, \mu_2, \ldots, \mu_d$ are zeros of $\psi(z^{-1})$ and $\kappa_1, \kappa_2, \ldots, \kappa_g$ are poles. The term z^{g-d} contributes a further g zeros and d poles at the origin. If these do not cancel completely, then they will leave, as a remainder, a positive or negative power of z whose phase-shifting effect has been mentioned above.

The BIBO stability condition requires that $\psi(z^{-1})$ must be finite-valued for all z with $|z| \geq 1$, for which it is necessary and sufficient that $|\kappa_j| < 1$ for all $j = 1, \ldots, g$.

The effect of the filter can be assessed by plotting its poles and zeros on an Argand diagram. The frequency-response function is simply the set of the values which are assumed by the complex function $\psi(z^{-1})$ as z travels around the unit circle; and, at any point on the circle, we can assess the contribution which each pole and zero makes to the gain and phase of the filter. Setting $z = e^{i\omega}$ in (2), which places z on the circumference of the unit circle, gives

$$\psi(e^{-i\omega}) = e^{i(g-d)\omega}\frac{\delta_0}{\gamma_0}\frac{(e^{i\omega}-\mu_1)(e^{i\omega}-\mu_2)\cdots(e^{i\omega}-\mu_d)}{(e^{i\omega}-\kappa_1)(e^{i\omega}-\kappa_2)\cdots(e^{i\omega}-\kappa_g)}. \tag{3}$$

The generic factors in this expression can be written in polar form as

$$\begin{aligned} e^{i\omega} - \mu_j &= |e^{i\omega} - \mu_j| e^{i\phi_j(\omega)} \\ &= \rho_j(\omega) e^{i\phi_j(\omega)} \end{aligned} \quad \text{and} \quad \begin{aligned} e^{i\omega} - \kappa_j &= |e^{i\omega} - \kappa_j| e^{i\varphi_j(\omega)} \\ &= \lambda_j(\omega) e^{i\varphi_j(\omega)}. \end{aligned} \tag{4}$$

When the frequency-response function as a whole is written in polar form, it becomes $\psi(\omega) = |\psi(\omega)|e^{-i\theta(\omega)}$, with

$$\begin{aligned} |\psi(e^{-i\omega})| &= \left|\frac{\delta_0}{\gamma_0}\right| \frac{|e^{i\omega}-\mu_1|\;|e^{i\omega}-\mu_2|\cdots|e^{i\omega}-\mu_d|}{|e^{i\omega}-\kappa_1|\;|e^{i\omega}-\kappa_2|\cdots|e^{i\omega}-\kappa_g|} \\ &= \left|\frac{\delta_0}{\gamma_0}\right| \frac{\prod \rho_j(\omega)}{\prod \lambda_j(\omega)}. \end{aligned} \tag{5}$$

and

$$\theta(\omega) = (d-g)\omega - \big\{\phi_1(\omega) + \cdots + \phi_d(\omega)\big\} + \big\{\varphi_1(\omega) + \cdots + \varphi_g(\omega)\big\}. \tag{6}$$

The value of $\lambda_j = |e^{i\omega} - \kappa_j|$ is simply the distance from the pole κ_j to the point $z = e^{i\omega}$ on the unit circle whose radius makes an angle of ω with the positive real axis. It can be seen that the value of λ_j is minimised when $\omega = \mathrm{Arg}(\kappa_j)$ and maximised when $\omega = \pi + \mathrm{Arg}(\kappa_j)$. Since λ_j is a factor in the denominator of the function $|\psi(\omega)|$, it follows that the pole κ_j makes its greatest contribution to the gain of the filter when $\omega = \mathrm{Arg}(\kappa_j)$ and its least contribution when $\omega = \pi + \mathrm{Arg}(\kappa_j)$. Moreover, if κ_j is very close to the unit circle, then its contribution to the gain at $\omega = \mathrm{Arg}(\kappa_j)$ will be very large. The effect of the zeros upon the gain of the filter is the opposite of the effect of the poles. In particular, a zero μ_j which lies on the perimeter of the unit circle will cause the gain of the filter to become zero at the frequency value which coincides with the zero's argument—that is to say, when $\omega = \mathrm{Arg}(\mu_j)$.

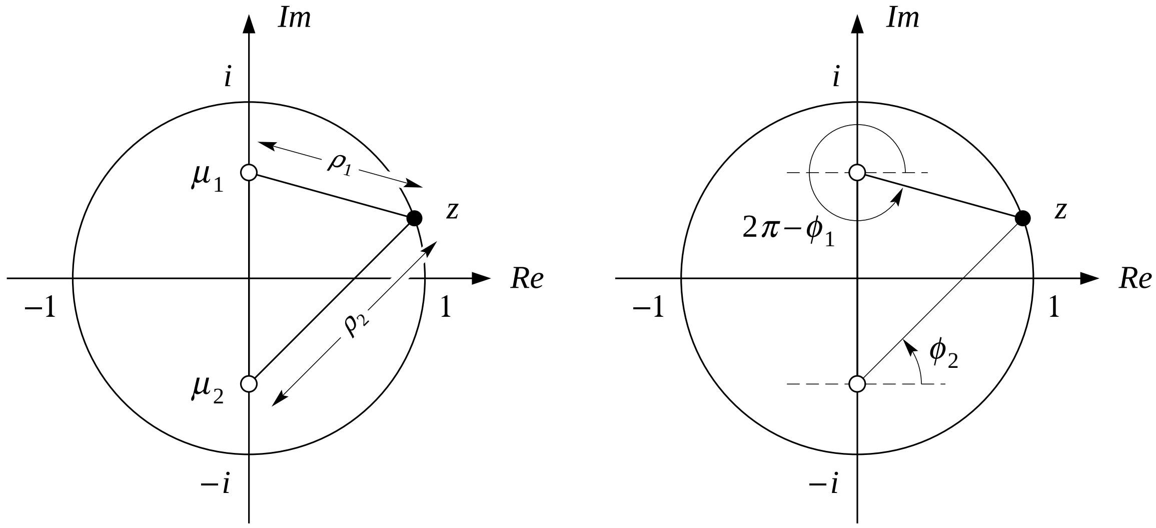

Figure 7. The pair of conjugate zeros $\mu_j, j = 1, 2$ are evaluated at the point $z = e^{i\omega}$ on the unit circle by finding the modulii $\rho_j = |z - \mu_j|$ and the arguments $\phi_j = \mathrm{Arg}(z - \mu_j)$ of the corresponding factors.

The effect of the placement of the poles and zeros upon the phase of the filter may also be discerned from the Argand diagram. Thus it can be seen that the value of the derivative $d\mathrm{Arg}(e^{i\omega} - \mu_j)/d\omega$ is maximised an the point on the circle where $\omega = \mathrm{Arg}(\mu_j)$. Moreover, the value of the maximum increases with the diminution of $|e^{i\omega} - \mu_j|$, which is the distance between the root and the point on the circle.

The results of this section may be related to the Argument Principle which declares that the number of times the trajectory of a function $f(z)$ encircles the origin as $z = e^{i\omega}$ travels around the unit circle is equal to the number N of the zeros of $f(z)$ which lie within the circle less the number P of the poles which lie within the circle.

In applying the principle in the present context, some account has to be taken of the fact that the polynomials comprised by the rational function $\psi(z^{-1})$ are in terms of negative powers of z. Thus, on factorising numerator polynomial $\delta(z^{-1}) = \delta_0 + \delta_1 z^{-1} + \cdots + \delta_d z^{-d}$, it is found that $\delta(z^{-1}) = \delta_0 \prod_j (1 - \mu_j/z) = z^{-d}\delta_0 \prod_j (z - \mu)$, which indicates that the polynomial contributes d poles as well as d zeros to the rational function.