SCALING AND MULTIRESOLUTION ANALYSIS

Each scaling function $\phi_{j,k}(x)$ has two indexes which are j, the dilation index, and k, the displacement index. The normalised function is given by

$$\phi_{j,k}(x) = c_j \phi\left(\frac{x - b_{j,k}}{a_j}\right) = 2^{-j/2}\phi(2^{-j}x - k), \tag{1}$$

where

$$\begin{aligned} a_j &= 2^j \quad \text{is the dispersion factor,} \\ b_{j,k} &= k a_j = 2^j k \quad \text{is the displacement,} \\ c_j &= 2^{-j/2} \quad \text{is a normalisation factor.} \end{aligned}$$

In the first expression on the RHS of (1), the argument $(x - b_{j,k})/a_j$ is reminiscent of the argument $(x - \mu)/\sigma$ of a normal distribution $N(\mu, \sigma^2)$, obtained by adjusting the value of x for its mean μ and dividing by the standard deviation σ.

Increasing values of j correspond to an increasing dilation or dispersion of the function $\phi_{j,k}(x)$. This can be understood in two ways. First, we may satisfy ourselves that the factor $a_j = 2^j$ bears a direct analogy with the standard deviation of a statistical distribution, such as the univariate normal distribution wherein the expression $-\frac{1}{2}(x - \mu)^2/\sigma^2$ is the quadratic exponent. Secondly, we may note that the argument $2^{-j}x$ varies more slowly with x as j increases. Hence the function $\phi_{j,k}(x)$ evolves more slowly the higher the values of j, which is to say that it becomes more dilated or dispersed.

In a dyadic multi resolution system, we can begin by defining a set of scaling functions

$$\{\phi(x - k) = \phi_{0,k}(x - k); k \in \mathcal{Z}\}, \tag{2}$$

which spans the frequency band $[0, \pi)$. The corresponding function space may be denoted by $\mathcal{V}_0$.

The dilation equation, which takes the form of

$$\phi(t) = 2^{1/2}\sum_k h_k \phi(2x - k), \tag{3}$$

defines the scaling function $\phi(x)$ in terms of a set of functions $\{\phi(2x - k); k \in \mathcal{Z}\}$ that have a lesser dispersion, and a wider frequency contents than those of (2). Notice that the dispersion factor on the RHS of (3) is $1/2 = a_{-1}$. This confirms that the scaling functions in the sum on the RHS are of lesser dispersion than the scaling function on the LHS. Therefore, it makes sense to denote the space of the less dispersed functions by $\mathcal{V}_{-1}$ and to write $\mathcal{V}_0 \subset \mathcal{V}_{-1}$, More generally, we should write

$$\cdots \subset \mathcal{V}_2 \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \mathcal{V}_{-2} \subset \cdots. \tag{4}$$

In the pyramid algorithm of Mallat, we proceed from space $\mathcal{V}_0$, which contains functions that are band-limited to the frequency interval $[0, \pi)$, to the included space $\mathcal{V}_1$, containing functions band-limited to $[0, \pi/2)$, and thence to the space $\mathcal{V}_2$, containing functions band-limited to $[0, \pi/4)$. In general, $\mathcal{V}_j$ contains functions band-limited to $[0, \pi/2^j)$. We can see that the index j is positively associated with *increasing dispersion* and *declining frequency.*

Other authors would associate the index j in this context with *increasing resolution* and, therefore, with *increasing frequency.* They would write $\mathcal{V}_1$ and $\mathcal{V}_2$ in place of $\mathcal{V}_{-1}$ and $\mathcal{V}_{-2}$, whence the inclusion relationships would become

$$\cdots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots. \tag{5}$$