THE SHANNON-WHITTAKER SAMPLING THEOREM

According to the Shannon–Whittaker sampling theorem, any square integrable piecewise continuous function $x(t) \longleftrightarrow \xi(\omega)$ that is band-limited in the frequency domain, with $\xi(\omega) = 0$ for $\omega > \pi$, has the series expansion

$$(1) \qquad x(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} x_k \psi_{(0)}(t-k),$$

where $x_k = x(k)$ is the value of the function $x(t)$ at the point $t = k$. It follow that the continuous function $x(t)$ can be reconstituted from its sampled values $\{x_t, t \in \mathcal{I}\}$.

Proof. Since $x(t)$ is a square-integrable function, it is amenable to a Fourier integral transform which gives

$$(2) \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega)e^{i\omega t}d\omega, \quad \text{where} \quad \xi(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt.$$

But $\xi(\omega)$ is a continuous function defined of the interval $(-\pi, \pi]$ that may also be regarded as a periodic function of a period of 2π. Therefore, $\xi(\omega)$ is amenable to a classical Fourier analysis; and it may be expanded as

$$(3) \qquad \xi(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{-ik\omega}, \quad \text{where} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega)e^{ik\omega}d\omega.$$

By comparing (2) with (3), we see that the coefficients c_k are simply the ordinates of the function $x(t)$ sampled at the integer points; and we may write them as

$$(4) \qquad c_k = x_k = x(k).$$

Next, we must show how the continuous function $x(t)$ may be reconstituted from its sampled values. Using (4) in (3) gives

$$(4) \qquad \xi(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-ik\omega}.$$

Putting this in (2), and taking the integral over $(-\pi, \pi]$ in consequence of the band-limited nature of the function $x(t)$, gives

$$(5) \quad x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=-\infty}^{\infty} x_k e^{-ik\omega}\right\}e^{i\omega t}d\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x_k \int_{-\pi}^{\pi} e^{i\omega(t-k)}d\omega.$$

The integral on the RHS is evaluated as

$$(6) \qquad \int_{-\pi}^{\pi} e^{i\omega(t-k)}d\omega = 2\frac{\sin\{\pi(t-k)\}}{t-k}.$$

Putting this into the RHS of (5) gives the result of (1).