FILTERING SHORT SEQUENCES

Filtering Stationary Sequences

Imagine that a short sequence of observations has been sampled from a process

$$y(t) = \xi(t) + \eta(t), \tag{1}$$

where $\xi(t)$ is a signal and $\eta(t)$ is a noise process which tends to obscure the signal. It is assumed that the processes $\xi(t)$ and $\eta(t)$ are stationary and mutually independent and that their statistical properties may be summarised by their first and second moments.

The set of observations can be are gathered in a vector

$$y = \xi + \eta. \tag{2}$$

It is assumed that

$$E(\xi) = 0 \qquad \text{and} \qquad D(\xi) = \sigma_\nu^2 \Omega_S, \tag{3}$$

and that

$$E(\eta) = 0 \qquad \text{and} \qquad D(\eta) = \sigma_\varepsilon^2 \Omega_N. \tag{4}$$

Hence, in view of the statisitical independence of $\xi(t)$ and $\eta(t)$, it follows

$$E(y) = 0 \qquad \text{and} \qquad D(y) = \sigma_\nu^2 \Omega_S + \sigma_\varepsilon^2 \Omega_N. \tag{5}$$

If both ξ and η are generated by moving-average processes, then Ω_S and Ω_N will be symmetric Toeplitz matrices with a limited number of nonzero diagonal bands.

The optimal predictor x of the vector ξ is given by the following conditional expectation:

$$\begin{aligned} E(\xi|y) &= E(\xi) + C(\xi, y)D^{-1}(y)\{y - E(y)\} \\ &= \Omega_S(\Omega_S + \lambda\Omega_N)^{-1}y = x, \end{aligned} \tag{6}$$

where $\lambda = \sigma_\varepsilon^2/\sigma_\nu^2$. The optimal predictor h of η is given, likewise, by

$$\begin{aligned} E(\eta|y) &= E(\eta) + C(\eta, y)D^{-1}(y)\{y - E(y)\} \\ &= \lambda\Omega_N(\Omega_S + \lambda\Omega_N)^{-1}y = h. \end{aligned} \tag{7}$$

It may be confirmed that $x + h = y$.

The estimates are calculated, first, by solving the equation

$$(\Omega_S + \lambda\Omega_N)b = y \tag{8}$$

for the value of b and, thereafter, by finding

$$x = \Omega_S b \qquad \text{and} \qquad h = \lambda\Omega_N b. \tag{9}$$

The solution of equation (8) is found via a Cholesky factorisation which sets $\Omega_S + \lambda\Omega_N = GG'$, where G is a lower-triangular matrix. The system $GG'b = y$ may be cast in the form of $Gp = y$ and solved for p. Then $G'b = p$ can be solved for b.

Filtering Nonstationary Sequences

Now consider the case where $y(t) = \xi(t) + \eta(t)$ is a nonstationary sequence comprising a nonstationary signal $\xi(t)$ and a stationary noise component $\eta(t)$. Imagine that d differences are sufficient to reduce $\xi(t)$ to stationarity, and let Q' be the matrix counterpart of the operator $(I - L)^d$ which produces $\zeta(t) = (I-L)^d\xi(t)$ and $\kappa(t) = (I-L)^d\eta(t)$, which are statistically independent processes. Then

$$\begin{aligned} Q'y &= Q'\xi + Q'\eta \\ &= \zeta + \kappa = g. \end{aligned} \tag{10}$$

It is assumed that

$$E(\zeta) = 0 \qquad \text{and} \qquad D(\zeta) = \sigma_\nu^2\Omega_S, \tag{11}$$

and that

$$E(\kappa) = 0 \qquad \text{and} \qquad \begin{aligned} D(\kappa) &= Q'D(\eta)Q \\ &= \sigma_\varepsilon^2 Q'\Sigma Q = \sigma_\varepsilon^2\Omega_N. \end{aligned} \tag{12}$$

The estimator z of the differenced signal is therefore

$$\begin{aligned} E(\zeta|g) &= E(\zeta) + C(\zeta, g)D^{-1}(g)\{g - E(g)\} \\ &= \Omega_S(\Omega_S + \lambda\Omega_N)^{-1}g = z, \end{aligned} \tag{13}$$

where $\lambda = \sigma_\varepsilon^2/\sigma_\nu^2$. The estimator k of the differenced noise vector κ is

$$\begin{aligned} E(\kappa|g) &= E(\kappa) + C(\kappa, g)D^{-1}(g)\{g - E(g)\} \\ &= \lambda\Omega_N(\Omega_S + \lambda\Omega_N)^{-1}g = k. \end{aligned} \tag{14}$$

The estimates are calculated, first, by solving the equation

$$(\Omega_S + \lambda\Omega_N)b = g \tag{15}$$

for the value of b and, thereafter, by finding

$$z = \Omega_S b \qquad \text{and} \qquad k = \lambda\Omega_N b. \tag{16}$$

In order to recover and estimate x of the trend ξ from the estimate z of the differenced vector $\zeta = Q'\xi$, we adopt the following criterion:

$$\text{Minimise} \quad (y - x)'\Sigma^{-1}(y - x) \quad \text{subject to} \quad Q'x = z. \tag{17}$$

This is a matter of finding an estimated trend vector which is closely aligned to the data and which has a differenced value equal to the filtered value z generated by equation (13). Therefore, we consider the Lagrangean function

$$L(x, \mu) = (y - x)'\Sigma^{-1}(y - x) + 2\mu'(Q'x - z). \tag{18}$$

By differentiating the function with respect to x and setting the result to zero, we obtain the condition

$$\Sigma^{-1}(y - x) - Q\mu = 0. \tag{19}$$

Premultiplying by $Q'\Sigma$ gives

$$Q'(y - x) = Q'\Sigma Q\mu. \tag{20}$$

But, from (15) and (16), it follows that

$$\begin{aligned} Q'(y - x) &= g - z \\ &= \lambda\Omega_R b = \lambda Q'\Sigma Q b, \end{aligned} \tag{21}$$

whence, from (20), we get

$$\begin{aligned} \mu &= (Q'\Sigma Q)^{-1}Q'(y - x) \\ &= \lambda b. \end{aligned} \tag{22}$$

Putting the final expression for μ into (19) gives

$$x = y - \lambda\Sigma Q b, \tag{23}$$

which is the equation for estimating the trend.