THE PERIODOGRAM AND THE CIRCULAR AUTOCOVARIANCES

A natural way of representing the serial dependence of the elements of the data sequence $[y_0, y_1, \ldots, y_{T-1}]$ is to estimate their autocovariances. The empirical autocovariance of lag τ is defined by the formula

$$(1)\qquad c_\tau = \frac{1}{T}\sum_{t=\tau}^{T-1}(y_t - \bar{y})(y_{t-\tau} - \bar{y}).$$

The empirical autocorrelation of lag τ is defined by $r_\tau = c_\tau / c_0$ where c_0, which is formally the autocovariance of lag 0, is the variance of the sequence. The autocorrelation provides a measure of the relatedness of data points separated by τ periods which is independent of the units of measurement.

The periodogram may be written as

$$(2)\quad I(\omega_j) = \frac{T}{2}\rho_j^2 = \frac{2}{T}\left[\left\{\sum_{t=0}^{T-1}\cos(\omega_j t)(y_t - \bar{y})\right\}^2 + \left\{\sum_{t=0}^{T-1}\sin(\omega_j t)(y_t - \bar{y})\right\}^2\right],$$

where $\omega_j = 2\pi j/T$ is the jth Fourier frequency, which relates to the trigonometrical function that completes j cycles in the period spanned by the data. We should also be aware of the identity $\sum_t \cos(\omega_j t)(y_t - \bar{y}) = \sum_t \cos(\omega_j t) y_t$, which follows from the fact that, by construction, $\sum_t \cos(\omega_j t) = 0$ for all j. The inclusion of $\bar{y}$ in (2) is to assist in the ensuing developmemts.

It is straightforward to establish the relationship between the periodogram and the sequence of autocovariances. Expanding the RHS of (2) gives

$$(3)\qquad \begin{aligned} I(\omega_j) =& \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j t)\cos(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\} \\ &+ \frac{2}{T}\left\{\sum_t\sum_s \sin(\omega_j t)\sin(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\}, \end{aligned}$$

and, by using the identity $\cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A - B)$, we can rewrite this as

$$(4)\qquad I(\omega_j) = \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j[t-s])(y_t - \bar{y})(y_s - \bar{y})\right\}.$$

Next, on defining $\tau = t - s$ and writing $c_\tau = \sum_t (y_t - \bar{y})(y_{t-\tau} - \bar{y})/T$, we can reduce the latter expression to

$$(5)\qquad I(\omega_j) = 2\sum_{\tau=1-T}^{T-1}\cos(\omega_j\tau)c_\tau = 2\left\{c_0 + 2\sum_{\tau=1}^{T-1}\cos(\omega_j\tau)c_\tau\right\},$$

which is a Fourier transform of the sequence of empirical autocovariances.

From the definition of the frequency ω_j, it follows that $\cos(\omega_j\{T-\tau\}) = \cos(\omega_j\tau)$, which is to say that the cosine is an even function of the index $\tau = 0,\ldots,T-1$. Therefore, (5) can be rewritten as

$$I(\omega_j) = 2\left\{c_0 + \sum_{\tau=1}^{T-1}\cos(\omega_j\tau)(c_\tau + c_{T-\tau})\right\}, \tag{6}$$

Here, the values $\tilde{c}_0 = c_0, \tilde{c}_\tau = c_\tau + c_{T-\tau}; \tau = 1,\ldots T-1$ constitute the so-called circular autocovariances.

It is easy to see that there is a one-to-one correspondence between the sequence of circular autocovariances $\tilde{c}_0,\ldots,\tilde{c}_{T-1}$ and the sequence of periodogram ordinates $I_0,\ldots,I_{T-1}$. We have already seen in (6) that

$$I_j = \frac{T}{2}\rho_j^2 = 2\sum_{\tau=0}^{T-1}\tilde{c}_\tau\cos(\omega_j\tau). \tag{7}$$

To show that, conversely,

$$\tilde{c}_\tau = \frac{1}{T}\sum_{j=0}^{T-1}I_j\cos(\omega_j\tau), \tag{8}$$

we may substitute into the latter the expression for I_j. The result should be an identity. Thus we find that

$$\begin{aligned}\tilde{c}_\tau &= \frac{2}{T}\sum_{j=0}^{T-1}\cos(\omega_j\tau)\left\{\sum_{\kappa=0}^{T-1}\tilde{c}_\kappa\cos(\omega_j\kappa)\right\}\\ &= \frac{2}{T}\sum_{\kappa=0}^{T-1}\tilde{c}_\kappa\sum_{j=0}^{T-1}\cos(\omega_j\kappa)\cos(\omega_j\tau).\end{aligned} \tag{9}$$

But the orthogonality relationships affecting the cosine functions at the Fourier frequencies imply that

$$\sum_{j=0}^{T-1}\cos(\omega_j\kappa)\cos(\omega_j\tau) = \begin{cases}0, & \text{if } \kappa \neq \tau;\\ \frac{T}{2}, & \text{if } \kappa = \tau.\end{cases} \tag{10}$$

Using these results in (9) reduces the RHS to $\tilde{c}_\tau$, which establishes the necessary identity.