STRUCTURAL TIME-SERIES MODELS

The structural time-series model represents the components of a time series, which are its trend, its cyclical components and its irregular component, as the products of independent ARIMA processes. The basic structural model, which lacks a non-seasonal cyclical component, takes the form of

$$y(t) = \lambda(t) + \sigma(t) + \varepsilon(t). \tag{1}$$

The trend or levels component $\lambda(t)$ is a stochastic process which generates a trajectory that is approximately linear within a limited locality. Thus

$$\lambda(t) = \lambda(t-1) + \beta(t-1) + \eta(t) \quad \text{or, equivalently,} \quad \nabla\lambda(t) = \beta(t-1) + \eta(t). \tag{2}$$

That is to say, the change in the level of the trend is compounded from the slope parameter $\beta(t-1)$, generated in the previous period, and a small white-noise disturbance $\eta(t)$. The slope parameter follows a random walk. Thus

$$\beta(t) = \beta(t-1) + \zeta(t) \quad \text{or, equivalently,} \quad \nabla\beta(t) = \zeta(t), \tag{3}$$

where $\zeta(t)$ denotes a white-noise process which is independent of the disturbance process $\eta(t)$. By applying the difference operator to equation (2) and substituting from (3), we find that

$$\begin{aligned} \nabla^2\lambda(t) &= \nabla\beta(t-1) + \nabla\eta(t) \\ &= \zeta(t-1) + \nabla\eta(t). \end{aligned} \tag{4}$$

The two terms of the RHS can be combined to form a first-order moving averages process, whereupon the process generating $\lambda(t)$ can be described by an integrated moving-average IMA(2, 1) model. Thus

$$\begin{aligned} \nabla^2\lambda(t) &= \zeta(t-1) + \nabla\eta(t) \\ &= (1-\mu L)\nu(t). \end{aligned} \tag{5}$$

A limiting case arises when the variance of the white-noise process $\zeta(t)$ in equation (3) tends to zero. Then the slope parameter tends to a constant β, and the process by which the trend is generated, which has been identified as an IMA(2,1) process, becomes a random walk with drift.

Another limiting case arises when the variance of $\eta(t)$ in equation (2) tends to zero. Then the overall process generating the trend becomes a second-order random walk, and the resulting trends are liable to be described as smooth trends.

When the variances of $\zeta(t)$ and $\eta(t)$ are both zero, then the process $\lambda(t)$ degenerates to a simple linear time trend.

The seasonal component of the structural time-series model is described by the equation

$$\begin{aligned} &\sigma(t) + \sigma(t-1) + \cdots + \sigma(t-s+1) = \omega(t) \\ &\text{or, equivalently,} \\ &S\sigma(t) = \omega(t), \end{aligned} \tag{6}$$

where $S = 1 + L + L^2 + \cdots + L^{s-1}$ is the seasonal summation operator, s is the number of observation per annum, and $\omega(t)$ is a white-noise process.

The equation implies that the sum of s consecutive values of this component will be a random variable distributed about a mean of zero. To understand this construction, we should note that, if the seasonal pattern were perfectly regular and invariant, then the sum of the consecutive values would be identically zero. Since the sum is a random variable with a zero mean, some variability can occur in the seasonal pattern.

By combining equations (1), (4) and (6), we seen that the structural model can be represented by the equation

$$\begin{aligned} &\nabla^2 Sy(t) = S\zeta(t-1) + \nabla S\eta(t) + \nabla^2\omega(t) + \nabla^2 S\varepsilon(t), \\ &\text{or, equivalently,} \\ &\nabla\nabla_s y(t) = S\zeta(t-1) + \nabla_s\eta(t) + \nabla^2\omega(t) + \nabla\nabla_s\varepsilon(t), \end{aligned} \tag{7}$$

where $\zeta(t)$, $\eta(t)$, $\omega(t)$ and $\varepsilon(t)$ are mutually independent white-noise processes. Here the alternative expression comes from using the identity $\nabla S = (1-L)(1+L+\cdots+L^{s-1}) = (1-L^s) = \nabla_s$. We should observe that the RHS or equation (7) constitutes a moving average of degree, $s+1$ which is typically subject to a number of restriction on its parameters. The restrictions arise from the fact there are only four parameters in the model of (7), which are the white noise variances $V\{\zeta(t-1)\}$, $V\{\eta(t)\}$, $V\{\omega(t)\}$ and $V\{\varepsilon(t)\}$, whereas there are $s+1$ moving average parameters and a variance parameter in the unrestricted reduced-form of the seasonal ARIMA model.

The basic structural model can be represented is a state-space form which comprises a transition equation, which constitutes a first-order vector autoregressive process, and an accompanying measurement equation. For notational convenience, let $s = 4$, which corresponds to the case of quarterly observations on annual data. Then the transition equation, which gathers together equations (2) (3) and (6), is

$$\begin{bmatrix} \lambda(t) \\ \beta(t) \\ \sigma(t) \\ \sigma(t-1) \\ \sigma(t-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda(t-1) \\ \beta(t-1) \\ \sigma(t-1) \\ \sigma(t-2) \\ \sigma(t-3) \end{bmatrix} + \begin{bmatrix} \eta(t) \\ \zeta(t) \\ \omega(t) \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

The observation equation, which corresponds to (1), is

$$y(t) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda(t) \\ \beta(t) \\ \sigma(t) \\ \sigma(t-1) \\ \sigma(t-2) \end{bmatrix} + \varepsilon(t). \tag{9}$$

The state-space model is amenable to the Kalman filter and the associated smoothing algorithms, which can be used in estimating the parameters of the model and in extracting estimates of the so-called unobserved components $\lambda(t)$, $\sigma(t)$ and $\varepsilon(t)$.