An Integrated Wiener Processes and its Discrete-Time Analogue

A Wiener process $Z(t)$ consists of an accumulation of independently distributed stochastic increments. The path of $Z(t)$ is continuous almost everywhere and differentiable almost nowhere. If $dZ(t)$ stands for the increment of the process in the infinitesimal interval dt, and if $Z(a)$ is the value of the function at time a, then the value at time $\tau > a$ is given by

$$Z(\tau) = Z(a) + \int_a^\tau dZ(t). \tag{1}$$

Moreover, it is assumed that the change in the value of the function over any finite interval $(a, \tau]$ is a random variable with a zero expectation:

$$E\{Z(\tau) - Z(a)\} = 0. \tag{2}$$

Let us write $ds \cap dt = \emptyset$ whenever ds and dt represent non-overlapping intervals. Then the conditions affecting the increments may be expressed by writing

$$E\{dZ(s)dZ(t)\} = \begin{cases} 0, & \text{if } ds \cap dt = \emptyset; \\ \sigma^2 dt, & \text{if } ds = dt. \end{cases} \tag{3}$$

These conditions imply that the variance of the change over the interval $(a, \tau]$ is proportional to the length of the interval. Thus

$$\begin{aligned} V\{Z(\tau) - Z(a)\} &= \int_{s=a}^\tau \int_{t=a}^\tau E\{dZ(s)dZ(t)\} \\ &= \int_{t=a}^\tau \sigma^2 dt = \sigma^2(\tau - a). \end{aligned} \tag{4}$$

The definite integrals of the Wiener process may be defined also in terms of the increments. The value of the first integral at time τ is given by

$$\begin{aligned} Z^{(1)}(\tau) &= Z^{(1)}(a) + \int_a^\tau Z(t)dt \\ &= Z^{(1)}(a) + Z(a)(\tau - a) + \int_a^\tau (\tau - t)dZ(t), \end{aligned} \tag{5}$$

where the second equality comes via (85). The mth integral is

$$Z^{(m)}(\tau) = \sum_{k=0}^{m} Z^{(m-k)}(a)\frac{(\tau - a)^k}{k!} + \int_a^\tau \frac{(\tau - t)^m}{m!} dZ(t). \tag{6}$$

The covariance of the changes $Z^{(j)}(\tau) - Z^{(j)}(a)$ and $Z^{(k)}(\tau) - Z^{(k)}(a)$ of the jth and the kth integrated processes derived from $Z(t)$ is given by

$$\begin{aligned} C_{(a,\tau)}\{z^{(j)}, z^{(k)}\} &= \int_{s=a}^\tau \int_{t=a}^\tau \frac{(\tau - s)^j(\tau - t)^k}{j!k!} E\{dZ(s)dZ(t)\} \\ &= \sigma^2 \int_a^\tau \frac{(\tau - t)^j(\tau - t)^k}{j!k!} dt = \sigma^2 \frac{(\tau - a)^{j+k+1}}{(j + k + 1)j!k!}. \end{aligned} \tag{7}$$

The object of our exercise is to find the form of a discrete-time model which will represent a sequence of observations $y_0, y_1, \ldots, y_n$ taken of an integrated Wiener process at the times $t_0, t_1, \ldots, t_n$. The interval between t_i and t_{i-1} is $h_i = t_i - t_{i-1}$ which, for the sake of generality, will be allowed to vary in the first instance, albeit that, ultimately, we shall set $h_i = 1$ for all i.

In order to conform to the existing notation, we define

$$(8) \qquad \zeta_i = Z(t_i) \qquad \text{and} \qquad \xi_i = Z^{(1)}(t_i)$$

to be, respectively, the slope of the trend component and its level at time t_i, where $Z(t_i)$ and $Z^{(1)}(t_i)$ are described by equations (1) and (5). Also we define

$$(9) \qquad \varepsilon_i = \int_{t_{i-1}}^{t_i} dZ(t) \qquad \text{and} \qquad \nu_i = \int_{t_{i-1}}^{t_i} (t_i - t) dZ(t).$$

Then the equation for the slope, which was

$$(10) \qquad Z(t_i) = Z(t_{i-1}) + \int_{t_{i-1}}^{t_i} dZ(t),$$

becomes

$$(11) \qquad \zeta_i = \zeta_{i-1} + \varepsilon_i,$$

and the equation for the level, which was

$$(12) \qquad Z^{(1)}(t_i) = Z^{(1)}(t_{i-1}) + Z(t_i)h_i + \int_{t_{i-1}}^{t_i} (t_i - t_{i-1}) dZ(t),$$

becomes

$$(13) \qquad \xi_i = \xi_{i-1} + \zeta_{i-1} h_i + \nu_i.$$

The model of the underlying trend can now be written is state-space form as follows:

$$(14) \qquad \begin{bmatrix} \xi_i \\ \zeta_i \end{bmatrix} = \begin{bmatrix} 1 & h_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_{i-1} \\ \zeta_{i-1} \end{bmatrix} + \begin{bmatrix} \nu_i \\ \varepsilon_i \end{bmatrix},$$

whilst a corresponding observation which associates an error η_i with the ith observation would be written as

$$(15) \qquad y_i = [1 \quad 0] \begin{bmatrix} \xi_i \\ \zeta_i \end{bmatrix} + \eta_i.$$

Using the result under (7), we find that the dispersion matrix for the state disturbances is

$$(16) \qquad D \begin{bmatrix} \nu_i \\ \varepsilon_i \end{bmatrix} = \sigma_\varepsilon^2 \begin{bmatrix} \frac{1}{3} h_i^3 & \frac{1}{2} h_i^2 \\ \frac{1}{2} h_i^2 & h_i \end{bmatrix},$$

where σ_ε^2 is the variance of the Wiener process.

To simplify matters we may assume that the time intervals between observations are constant with $h_i = 1$ for all i. The the processes generating the sequences $\{\zeta_t\}$ and $\{\xi_t\}$ can be written as

$$(17)\qquad \begin{aligned} \xi(t) &= \xi(t-1) + \zeta(t-1) + \nu(t), \qquad \text{or} \\ (I-L)\xi(t) &= \zeta(t-1) + \nu(t), \end{aligned}$$

and

$$(18)\qquad \begin{aligned} \zeta(t) &= \zeta(t-1) + \varepsilon(t), \qquad \text{or} \\ (I-L)\zeta(t) &= \varepsilon(t). \end{aligned}$$

Combining the two equations gives

$$(19)\qquad \begin{aligned} \xi(t) &= \frac{\zeta(t-1)}{I-L} + \frac{\nu(t)}{I-L} \\ &= \frac{\varepsilon(t-1)}{(I-L)^2} + \frac{\nu(t)}{I-L}. \end{aligned}$$

or equivalently

$$(20)\qquad \begin{aligned} (I-L)^2\xi(t) &= \varepsilon(t-1) + (I-L)\nu(t) \\ &= \nu(t) - \nu(t-1) + \varepsilon(t-1). \end{aligned}$$

On the RHS of this equation is a sum of stationary stochastic process which can be expressed as an ordinary first-order moving-average process. The parameters of the latter process may be inferred from it autocovariances which arise from a combination of the autocovariances of $\varepsilon(t)$ and $\nu(t)$. The variance γ_0 of the MA process is given by the sum of the elements of the matrix

$$(21)\qquad E\begin{bmatrix} \nu_t^2 & -\nu_t\nu_{t-1} & \nu_t\varepsilon_{t-1} \\ -\nu_{t-1}\nu_t & \nu_{t-1}^2 & -\nu_{t-1}\varepsilon_{t-1} \\ \varepsilon_{t-1}\nu_t & -\varepsilon_{t-1}\nu_{t-1} & \varepsilon_{t-1}^2 \end{bmatrix} = \sigma_\varepsilon \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Thus it is found that $\gamma_0 = 4\sigma_\varepsilon/6$ The first autocovariance γ_1 of the MA process is given by the sum of the elements of the matrix

$$(22)\qquad E\begin{bmatrix} \nu_t\nu_{t-1} & -\nu_t\nu_{t-2} & \nu_t\varepsilon_{t-2} \\ -\nu_{t-1}^2 & \nu_{t-1}\nu_{t-2} & -\nu_{t-1}\varepsilon_{t-2} \\ \varepsilon_{t-1}\nu_{t-1} & -\varepsilon_{t-1}\nu_{t-2} & \varepsilon_{t-1}\varepsilon_{t-2} \end{bmatrix} = \sigma_\varepsilon \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Thus $\gamma_1 = \sigma_\varepsilon/6$.