Computing the MA Parameters from the Autocovariances

Consider the following equations which define the autocovaraiances of an MA(q) proccess:

$$\begin{aligned}
\gamma_0 &= \sigma_\varepsilon^2(\mu_0^2 + \mu_1^2 + \cdots + \mu_q^2),\\
\gamma_1 &= \sigma_\varepsilon^2(\mu_0\mu_1 + \mu_1\mu_2 + \cdots + \mu_{q-1}\mu_q),\\
&\vdots\\
\gamma_q &= \sigma_\varepsilon^2\mu_0\mu_q.
\end{aligned} \tag{1}$$

These equations define a mapping from the set of parameters to the sequence of autocovariances. If none of the roots of the polynomial equation $\mu(z) = 0$ lie on the unit circle, then the equations also serve to define an inverse mapping from the autocovariances to a set of parameters which correspond to a unique stationary process that satisfies the condition of invertibility. The equations which we must solve to obtain these parameters can be written, in two alternative ways, as

$$\begin{aligned}
\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{q-1} \\ \gamma_q \end{bmatrix} &= \sigma_\varepsilon^2 \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{q-1} & \mu_q \\ \mu_1 & \mu_2 & \cdots & \mu_q & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{q-1} & \mu_q & \cdots & 0 & 0 \\ \mu_q & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{q-1} \\ \mu_q \end{bmatrix}\\
&= \sigma_\varepsilon^2 \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{q-1} & \mu_q \\ 0 & \mu_0 & \cdots & \mu_{q-2} & \mu_{q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mu_0 & \mu_1 \\ 0 & 0 & \cdots & 0 & \mu_0 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{q-1} \\ \mu_q \end{bmatrix}
\end{aligned} \tag{2}$$

or, in summary notation, as

$$\gamma = \sigma_\varepsilon^2 M^{\#}\mu = \sigma_\varepsilon^2 M'\mu. \tag{3}$$

Therefore our objective is to find a solution, in terms of the vector μ, for the system

$$f(\mu) = \gamma - \sigma_\varepsilon^2 M^{\#}\mu = 0. \tag{4}$$

Since this system is nonlinear, we have to find the solution via an iterative procedure. If we use the Newton–Raphson procedure, then we derive the following

algorithm by which μ_{r+1}, which is the $(r+1)$th approximation to the solution, is obtained from the rth approximation μ_r:

$$\mu_{r+1} = \mu_r - \{Df(\mu_r)\}^{-1} f(\mu_r). \tag{5}$$

Here $f(\mu_r)$ and $Df(\mu_r)$ stand respectively for the vector function and its matrix first derivative evaluated at the point $\mu = \mu_r$. We can easily verify that

$$Df(\mu) = -\sigma_\varepsilon^2 (M^\# + M'). \tag{6}$$

Therefore the algorithm can be written as

$$\mu_{r+1} = \mu_r + \left\{\sigma_\varepsilon^2 (M^\# + M')\right\}_r^{-1} (\gamma - \sigma_\varepsilon^2 M^\# \mu)_r, \tag{7}$$

where the subscript on the LHS is to indicate that the elements are to be evaluated at $\mu = \mu_r$.

The procedure requires some starting values. Let us recall that we can normalise the equation of the moving average either by setting $\sigma_\varepsilon^2 = 1$ or by setting $\mu_0 = 1$. If we set $\sigma_\varepsilon^2 = 1$, then it is reasonable to begin the iterations with $\mu_0 = \gamma_0$ and $\mu_1 = \mu_2 = \ldots = \mu_q = 0$. Once the iterative procedure has converged, we can renormalise the equation so that $\mu_0 = 1$. Our system of equations has multiple solutions. Nevertheless, our selection of starting values makes it virtually certain that the iterative procedure will converge upon the parameter values that correspond to the uniquely defined invertible MA model.