WIENER–KOLMOGOROV FILTERS

Imagine that a sequence of observations $y(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ on a stationary stochastic signal $\xi(t)$ are afflicted by errors which form a sequence $\eta(t)$ that is statistically independent of the signal. Then

$$y(t) = \xi(t) + \eta(t) \tag{1}$$

and, if the autocovariance functions of these processes are known, then there is scope for deriving better estimates of the signal than those which are provided by the unprocessed observations. In that case, an estimate of $\xi(t)$ may be obtained by filtering $y(t)$ to give

$$x(t) = \beta(L)y(t). \tag{2}$$

The coefficients of the optimal linear signal-extraction filter $\beta(L) = \{\cdots + \beta_{-1}L^{-1} + \beta_0 + \beta_1 L + \cdots\}$ are estimated by invoking the minimum-mean-square-error criterion. The errors in question are the elements of the sequence $e(t) = \xi(t) - x(t)$, where $x(t)$ is given by equation (2). The principle of orthogonality, by which the criterion is fulfilled, indicates that the errors must be uncorrelated with the elements in the information set $\mathcal{I}_t = \{y_{t-k}; k = 0, \pm 1, \pm 2, \ldots\}$. Thus

$$\begin{aligned} 0 &= E\Big\{y_{t-k}(\xi_t - x_t)\Big\} \\ &= E(y_{t-k}\xi_t) - \sum_j \beta_j E(y_{t-k}y_{t-j}) \\ &= \gamma_k^{y\xi} - \sum_j \beta_j \gamma_{k-j}^{yy}, \end{aligned} \tag{3}$$

for all k. The equation may be expressed, in terms of the z transform, as

$$\gamma^{y\xi}(z) = \gamma^{yy}(z)\beta(z), \tag{4}$$

where $\beta(z)$ stands for an indefinite two-sided Laurent series comprising both positive and negative powers of z.

Given the assumption that the elements of the noise sequence $\eta(t)$ are independent of those of the signal $\xi(t)$, it follows that

$$\gamma^{yy}(z) = \gamma^{\xi\xi}(z) + \gamma^{\eta\eta}(z) \qquad \text{and} \qquad \gamma^{y\xi}(z) = \gamma^{\xi\xi}(z). \tag{5}$$

It follows from (4) that the signal-extraction filter is

$$\beta_S(z) = \frac{\gamma^{\xi\xi}(z)}{\gamma^{yy}(z)} = \frac{\gamma^{\xi\xi}(z)}{\gamma^{\xi\xi}(z) + \gamma^{\eta\eta}(z)}. \tag{6}$$

The noise-extraction filter is just the complementary filter

$$\beta_N(z) = 1 - \beta_S(z) = \frac{\gamma^{\eta\eta}(z)}{\gamma^{\xi\xi}(z) + \gamma^{\eta\eta}(z)}. \tag{7}$$

To provide some results of a more specific nature, let us assume that the signal $\xi(t)$ is generated by an autoregressive moving-average process such that $\phi(L)\xi(t) = \theta(L)\nu(t)$, where $\nu(t)$ is a white-noise sequence with $V\{\nu(t)\} = \sigma_\nu^2$. Also, let the variance of the white-noise error process be denoted by $V\{\eta(t)\} = \sigma_\eta^2$. Then

$$y(t) = \frac{\theta(L)}{\phi(L)}\nu(t) + \eta(t), \tag{8}$$

whence

$$\gamma^{\xi\xi}(z) = \sigma_\nu^2 \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})} \quad \text{and} \quad \gamma^{yy}(z) = \sigma_\nu^2 \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})} + \sigma_\eta^2. \tag{9}$$

It follows from (6) that

$$\beta(z) = \frac{\sigma_\nu^2\theta(z)\theta(z^{-1})}{\sigma_\nu^2\theta(z)\theta(z^{-1}) + \sigma_\eta^2\phi(z)\phi(z^{-1})} = \frac{\sigma_\nu^2}{\sigma_\varepsilon^2}\frac{\theta(z)\theta(z^{-1})}{\mu(z)\mu(z^{-1})}. \tag{10}$$

Here the denominator corresponds to the autocovariance generating function of a synthetic moving-average process

$$\mu(L)\varepsilon(t) = \theta(L)\nu(t) + \phi(L)\eta(t). \tag{11}$$

The autocovariances of this process are obtained by adding the autocovariances of the constituent processes on the RHS. Then the Cramér–Wold factorisation is employed to find the coefficients of $\mu(z)$. The factors of the numerator of $\beta(z)$ are already known.

It is notable that the signal-extraction filter of (10) would also transpire if equation (8), which denotes the underlying model for the process $y(t)$, were replaced by the equation

$$y(t) = \theta(L)\nu(t) + \phi(L)\eta(t), \tag{12}$$

which comprises two independent moving-average processes.

In order to realise the filter $\beta(L)$, it is necessary to factorise it into two parts. The first part, which incorporates positive powers of the lag operator, runs forwards in time in the usual fashion. The second part of the filter, which incorporates negative powers of the lag operator, runs in reversed time.

Given this factorisation, the sequence $x(t)$, which estimates $\xi(t)$, can be found via two operations which are represented by

$$z(t) = \frac{\theta(L)}{\mu(L)}y(t) \quad \text{and} \quad x(t) = \frac{\theta(F)}{\mu(F)}z(t), \tag{13}$$

where $F = L^{-1}$ stands for the forward-shift operator whose effect is described by the equation $Fz(t) = z(t + 1)$. The reversed-time filtering which converts $z(t)$ into $x(t)$ is analogous to the smoothing operation which is associated with the Kalman filter. Taken together, the two filtering operations will have no net phase effect.